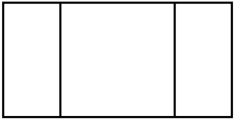
Workout 8 Solutions

Peter S. Simon

Homework: February 9, 2005

Problem 1

The measures of the length and width of a rectangular garden are each an integer. Ophelia fences the perimeter of the garden and also builds two divider fences, as shown. Numerically, the total length of fencing required in meters and the area of the garden in square meters are equal. What is the greatest possible area of this garden?



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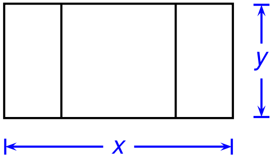

Call the dimensions of the garden x and y, as shown. Then

$$2x + 4y = xy \implies \frac{2}{y} + \frac{4}{x} = 1 \qquad (1)$$

$$\frac{2}{y} = 1 - \frac{4}{x} = \frac{x-4}{x} \implies \frac{y}{2} = \frac{x}{x-4} \implies y = \frac{2x}{x-4} \qquad (2)$$

x and y are both positive integers. From (1) we see that we must have $x > 4$ and $y > 2$ (why?). From (2) we can make a table of (x, y, xy) values for $x = 5, 6, 7, \ldots$. How far should we go in x?

Problem 1, Continued

How far should we go in x?

$$1 - \frac{4}{x} = \frac{2}{y} \leq \frac{2}{3} \implies \frac{4}{x} \geq 1 - \frac{2}{3} = \frac{1}{3} \implies \frac{x}{4} \leq 3 \implies x \leq 12$$

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x	$y = \frac{2x}{x-4}$	**Area** $= xy$
5	10	50
6	6	36
7	$4\frac{2}{3}$	N/A
8	4	32
9	$3\frac{3}{5}$	N/A
10	$3\frac{1}{3}$	N/A
11	$3\frac{1}{7}$	N/A
12	3	36

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Only 4 values of x result in integer values for y. The greatest area occurs when $x = 5$ and $y = 10$: Max. area $= \boxed{50}$.

Problem 2

On the last day of each calendar month, Sue pays off \$10 of her credit card balance. However, if there is still a balance remaining after this payment, a \$2 service fee is added to the remaining balance. On April 30, Sue paid off \$10 of her \$100 balance. If Sue never makes another charge to her card, at the beginning of what month will her balance first be below \$70?

Problem 2

On the last day of each calendar month, Sue pays off $10 of her credit card balance. However, if there is still a balance remaining after this payment, a $2 service fee is added to the remaining balance. On April 30, Sue paid off $10 of her $100 balance. If Sue never makes another charge to her card, at the beginning of what month will her balance first be below $70?

Date	Balance ($)
30 April	90
1 May	92
31 May	82
1 June	84
30 June	74
1 July	76
31 July	66
1 August	66

Problem 3

A clock on the mantle strikes one tone at 1:00 (am or pm), two tones at 2:00, three tones at 3:00 and so on. The first tone of the day is the first of 12 tones at midnight. How many tones will the clock strike in the month of January?

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The clock will strike 12 tones at midnight, 1 at 1:00 am, 2 at 2:00 am, . . . 11 at 11:00 am. It will then repeat this pattern: 12 at 12:00 pm, 1 at 1:00 pm, 2 at 2:00 pm, . . . , 11 at 11:00 pm. So in a single day, the clock strikes

$$2 \times (1 + 2 + 3 + \cdots + 12) = 2T_{12} = 2 \times \frac{12 \times 13}{2} = 12 \times 13 = 156$$

times. In January there are 31 days, so the clock will strike

$$31 \times 156 = \boxed{4836}$$

times.

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$$(2x + 1)^2 + (3x - 13)^2 = (3x - 4)^2$$

$$(4x^2 - 4x + 1) + (9x^2 - 78x + 169) = 9x^2 - 24x + 16$$

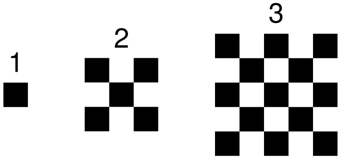
$$4x^2 - 58x + 154 = 0$$

$$2x^2 - 29x + 77 = 2(x - 11)(x - 3.5) = 0$$

for which the solutions are $x = 11$ or $x = 3.5$. But $x > \frac{13}{3} = 4\frac{1}{3}$ so that $x = \boxed{11}$ is the only solution.

Problem 5

If this pattern of black squares is continued, how many black squares will be in the 20th formation?



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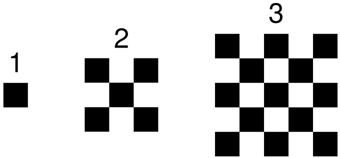

Formation #	# of Squares
1	1
2	$1 + 4$
3	$1 + 4 + 8$
4	$1 + 4 + 8 + 12$
$\vdots$	$\vdots$
n	$1 + 4 + 8 + 12 + \cdots + 4(n-1)$
$\vdots$	$\vdots$
20	$1 + 4 + 8 + \cdots + 4(19)$

Problem 5, Continued

So the number of squares in the 20th formation is

$$\begin{aligned} N &= 1 + 4 + 8 + 12 + \cdots + 4 \times 19 \\ &= 1 + 4(1 + 2 + 3 + \cdots + 19) \\ &= 1 + 4T_{19} \end{aligned}$$

where T_{19} is the 19th triangular number:
$T_n = 1 + 2 + \cdots + n = n(n+1)/2$, so that

$$\begin{aligned} N &= 1 + 4T_{19} \\ &= 1 + 4 \times \frac{19 \times 20}{2} \\ &= 1 + 2 \times 19 \times 20 = \boxed{761} \end{aligned}$$

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Try making a table (the Table feaure of your graphing calculator can help here):

x	$5^x - x^5$
1	4
2	−7
3	−118
4	−399
5	0
6	7849
7	61,318

so the correct answer is $x = \boxed{7}$.

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The largest product of N positive integers with a fixed sum is obtained when the integers are as nearly equal as possible. For example, if we consider $N = 2$, we see that

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So we can fill in the following table:

N	Largest Product of N Factors Whose Sum is 17
2	$8 \times 9 = 72$
3	$6 \times 6 \times 5 = 180$
4	$4 \times 4 \times 4 \times 5 = 320$
5	$3 \times 3 \times 3 \times 3 \times 5 = 405$
6	$3 \times 3 \times 3 \times 3 \times 3 \times 2 = 486$

Problem 7, Continued

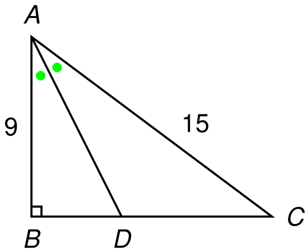
It turns out that

$$3 \times 3 \times 3 \times 3 \times 3 \times 2 = \boxed{486}$$

is the maximum product we can obtain. You can see this by noting that replacing a product of two 3s with three 2s (both add up to 6) will decrease the product, since $3 \times 3 = 9 > 2 \times 2 \times 2 = 8$.

Problem 8

Right triangle ABC is divided, as shown, such that angle BAC is bisected by $\overline{AD}$. What is the length of $\overline{AD}$? Express your answer as a decimal to the nearest hundredth.



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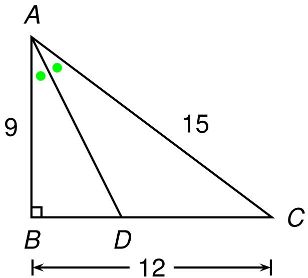

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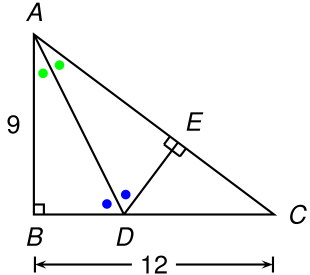

First, we note that $BC = 12$ since ABC is a 3-4-5 right triangle. Draw segment $\overline{DE}$ perpendicular to $\overline{AC}$. Since $\angle BAD = \angle DAE$ and the triangles share a hypotenuse, then triangles BAD and DAE are congruent,

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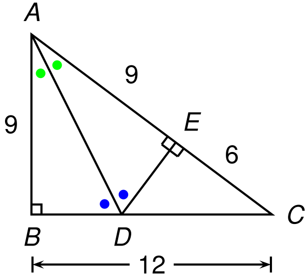

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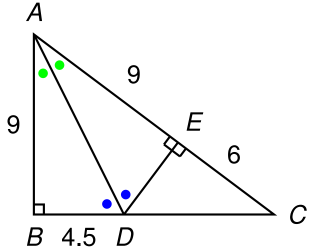

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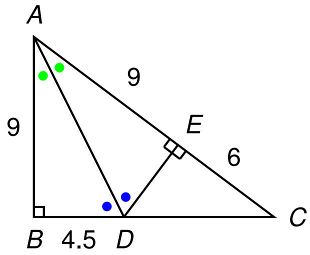

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Invoking Pythagoras one more time yields

$$AD = \sqrt{9^2 + 4.5^2} = \sqrt{101.25} \approx \boxed{10.06}$$

Problem 9

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Let x and y be the side dimensions of the rectangle. The diagonal length is then $D = \sqrt{x^2 + y^2}$. We are told that

$$\begin{aligned}
D = \sqrt{x^2 + y^2} &= x + y - 120 \\
x^2 + y^2 &= [(x + y) - 120]^2 = (x + y)^2 - 240(x + y) + 14400 \\
&= x^2 + 2xy + y^2 - 240(x + y) + 14400 \\
0 &= 2xy - 240(x + y) + 14400 \\
0 &= xy - 120(x + y) + 7200 = 48000 - 120(x + y) + 7200 \\
&= 55200 - 120(x + y) \\
0 &= 460 - (x + y) \implies (x + y) = 460
\end{aligned}$$

Problem 9, Continued

so

$$D = x + y - 120 = 460 - 120 = \boxed{540}$$

Problem 10

A box contains six cards. Three of the cards are black on both sides, one card is black on one side and red on the other, and two of the cards are red on both sides. If you pick a card at random from the box and see that the side facing you is red, what is the probability that the other side is red? Express your answer as a common fraction.

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Method 1

There are five different red sides that might be the one that you are looking at. Four of those sides have red on the other side of the card, too. Only one has black on the other side. The probability is $\boxed{\frac{4}{5}}$ that the other side is red, too.

Problem 10, Continued

Method 2: Conditional Probability

We use the formula for conditional probability: The probability of event a given that event b occurs is

$$P(a|b) = \frac{P(\text{Both } a \text{ and } b \text{ occur})}{P(b)} = \frac{P(a \cap b)}{P(b)}.$$

In this case a is the event that the second side of the card is red, and b is the event that the first side of the card is red. We calculate that

$$P(a \cap b) = P(\text{red/red card}) = \frac{2}{6} = \frac{1}{3}$$

$$P(b) = P(\text{red/red}) + P(\text{red/black, red examined 1st}) = \frac{2}{6} + \frac{1}{6} \cdot \frac{1}{2} = \frac{5}{12}$$

$$P(a|b) = \frac{\frac{1}{3}}{\frac{5}{12}} = \frac{1}{3} \times \frac{12}{5} = \boxed{\frac{4}{5}}$$