Workout 6 Solutions

Peter S. Simon

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Problem 1

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$$\begin{array}{cccc} & P & 4 & Q \\ + & 2 & R & 5 \\ \hline & S & 1 & T \end{array}$$

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Problem 3

A 6-ounce can of tuna has 13 grams of protein. This is 23% of the recommended daily allowance (RDA) based on a 2000-calorie diet. What is the number of grams of protein needed to get 100% of the RDA for this diet? Express your answer as a decimal to the nearest tenth.

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$$R = 13 \times \frac{100}{23} = \frac{1300}{23} \approx \boxed{56.5}$$

Problem 4

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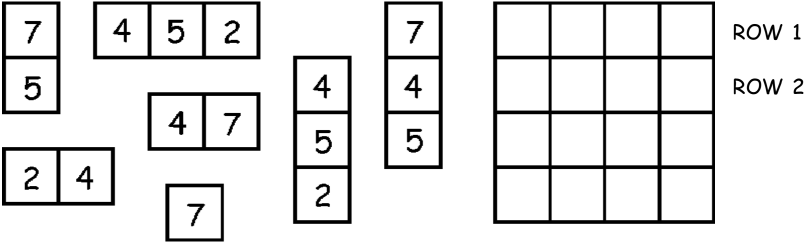
To use the minimum number of buses, as many 45-passenger busses as possible are filled first:

$$\frac{500}{45} = 11\tfrac{1}{9}$$

so that $\boxed{11}$ 45-passenger busses are used.

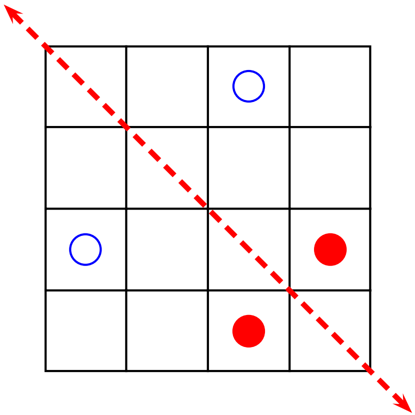
Problem 5

Without overlap, place these seven tiles into the 4×4 grid so that the four-digit number formed in the nth row equals the four-digit number in the nth column. The tiles may not be flipped or rotated. What is the sum of the four-digit numbers in Rows 1 and 2?

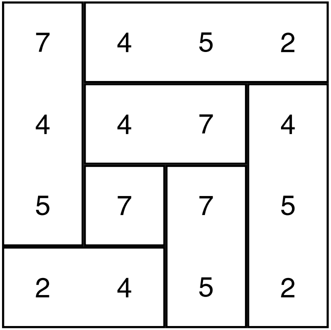


Problem 5, Continued

The key to solving this problem is to realize that we want to fill the matrix so that it is symmetrical about the red dashed line:



Problem 5, Continued



The sum of the two four-digit numbers in the top row is
$7452 + 4474 = \boxed{11{,}296}$

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If the ten letters in MATHCOUNTS are arranged at random, what is the probability that the word MATCH will be spelled out anywhere in the arrangement? Express your answer as a common fraction.

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There are ${}_{10}P_{10} = 10!$ ways to rearrange a 10-letter word. But since there are two “T” letters, so there are only $10!/2$ different words that can be formed from MATHCOUNTS.

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Now let's count the number of ways to spell MATCH. There are 6 locations that MATCH could take in the 10 letters:

MATCHxxxxx   xMATCHxxxx   xxMATCHxxx   xxxMATCHxx
xxxxMATCHx   xxxxxMATCH

For each of these 6 choices, there are ${}_{5}P_{5} = 5!$ ways to rearrange the other five letters, so there are a total of $6 \times 5! = 6!$ ways to form MATCH.

Problem 6, Continued

$$\text{Probability} = \frac{\text{\# Successful Rearrangements}}{\text{Total \# Rearrangements}}$$

$$= \frac{6!}{10!/2} = \frac{2}{10 \times 9 \times 8 \times 7}$$

$$= \boxed{\frac{1}{2520}}$$

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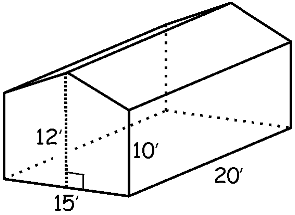
$$n^3 = (n+9)^2 - 9 = n^2 + 18n + 81 - 9 = n^2 + 18n + 72$$

Although there are formulas for solving cubic equations, they are very complicated. It's easier to just make a table:

n	n^3	$n^2 + 18n + 72$
1	1	91
2	8	112
3	27	135
4	64	160
5	125	187
$\boxed{6}$	216	216

Problem 8

A building has a rectangular foundation that is 20 × 15 feet. The roof extends between two opposite walls, is 12 feet high at the peak and slants down to 10 feet at the walls. What is the volume of the building?



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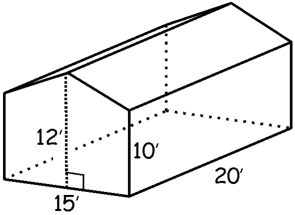

Note that the building is in the form of a cylinder, i.e., a solid figure obtained by sweeping a planar polygon (the base) along a line. The volume of any cylinder can be found by this formula:

$$\text{Cylinder Volume} = \text{Area of Base} \times \text{Cylinder Height}$$

$$V = Ah$$

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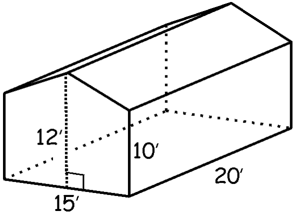

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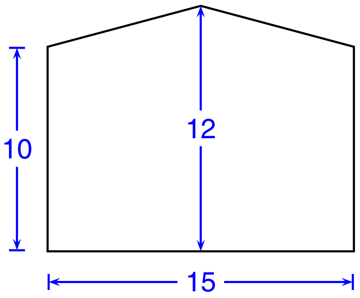
In this case the height of the cylinder is $h = 20$ feet.

Problem 8, Continued

The base can be decomposed into a triangle and rectangle.



The volume of the solid is then

$$V = Ah = 165 \times 20 = \boxed{3300\,\text{ft}^3}$$

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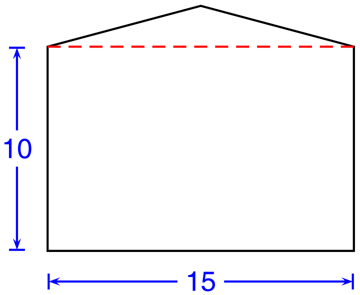
The base can be decomposed into a triangle and rectangle.

$$\text{Rectangle Area} = 10 \times 15 = 150$$

$$\text{Triangle Area} = \frac{1}{2} \times 2 \times 15 = 15$$

$$\text{Total Area} = A = 165$$



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let's list all the even, four-digit numbers containing only two different digits, the sum of whose digits is 9:

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Now divide each of these numbers by 2:

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Only $\boxed{2232}$ satisfies all the criteria.

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Steve could obtain 35 cents only by drawing one quarter and 2 nickels.

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$$p(\text{QNN}) = \frac{3}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{28}$$

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$$p(\text{QNN}) = \frac{3}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{28}$$

and similarly, the probability is also $1/28$ to draw NQN or to draw NNQ. So the probability that any of these three events occur is

$$3 \times \frac{1}{28} = \boxed{\frac{3}{28}}$$