Warm-Up 9 Solutions

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Problem 1

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In an arithmetic sequence, each term differs from its predecessor by a fixed amount. So

$$y - 12 = 12 - (y + 6)$$

$$y + (y + 6) = 12 + 12 = 24$$

$$2y = 24 - 6 = 18$$

$$y = \boxed{9}$$

Problem 2

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For Bobby to average 80 on five tests, he must score a total of $80 \times 5 = 400$ points. So on the last two tests he must score

$$400 - (73 + 83 + 90) = 400 - 246 = 154$$

So his average score on the last two tests must be at least

$$\frac{154}{2} = \boxed{77}$$

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Let N be the number of marbles in Anna's original collection. We are told that

$$\frac{N}{2} - 3 = 28$$
$$\frac{N}{2} = 31$$
$$N = \boxed{62}$$

Problem 4

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A convenient formula for a line given two points (x_1, y_1) and (x_2, y_2) is $y - y_1 = m(x - x_1)$, where m is the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{5 - 9} = -1$$

so the line is

$$y - 1 = -1(x - 9) = 9 - x \implies y = 10 - x$$

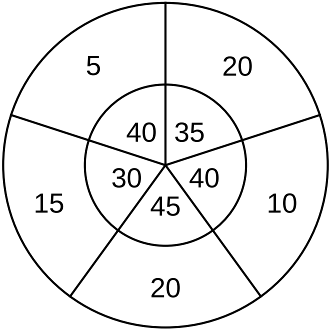
When $y = 0$ then

$$10 - x = 0 \implies x = 10$$

so the intersection point is $\boxed{(10, 0)}$

Problem 5

Three darts thrown at this dart board land in three different regions. If the order that the three darts are thrown does not matter, how many combinations of three different regions from the 10 possible regions would result in a sum of exactly 50 points?



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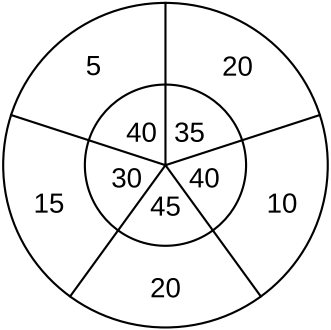

There are only $\boxed{3}$ ways to make 50 points using three different regions:

$$50 = 5 + 10 + 35$$

$$50 = 5 + 15 + 30$$

$$50 = 10 + 20 + 20$$

Problem 6

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For a year with 365 days, there will be 7 different calendars, corresponding to beginning the year on Monday, Tuesday, . . . , Saturday, or Sunday. There will be an additional 7 calendars required for a leap year. So the total number of calendars required is $\boxed{14}$.

Problem 7

Kris starts his run at 7 a.m. He runs at a rate of 4 miles per hour. His course takes him three miles out and then back along the same path. John runs at a rate of 6 miles per hour, starting at 7:10 a.m., and he runs the same course. How many minutes elapse between the time John finishes and the time Kris finishes?

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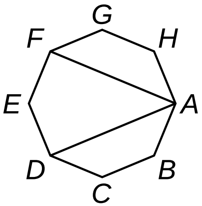
$$\text{Kris' Elapsed Time: } \frac{6\,\text{mi}}{4\,\text{mi/hr}} = 1.5\,\text{hr}$$

$$\text{John's Elapsed Time: } \frac{6\,\text{mi}}{6\,\text{mi/hr}} = 1\,\text{hr}$$

If they started together, John would finish 30 min ahead of Kris. Since he started 10 min later than Kris, John finished $\boxed{20\,\text{min}}$ ahead of Kris.

Problem 8

In the regular octagon *ABCDEFGH*, what is the number of degrees in the measure of angle *DAF*?



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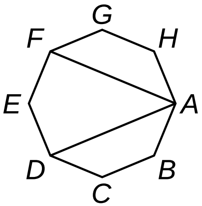
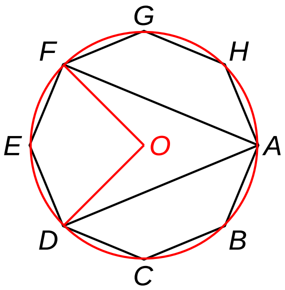


The easiest way to proceed is to circumscribe the octagon with a circle and note that the central angle *DOF* measures

$$m\angle DOF = 2 \times \frac{360^\circ}{8} = 90^\circ$$

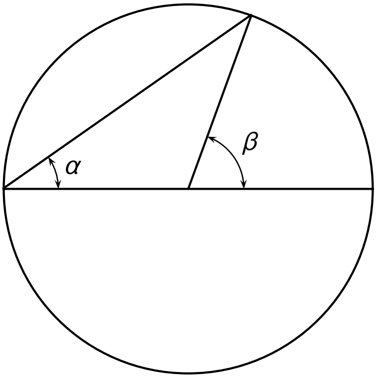


The inscribed angle is one-half of the central angle so

$$m\angle DAF = \frac{1}{2} m\angle DOF = \boxed{45^\circ}$$

Circles: Central and Inscribed Angles

First we examine a special case where one edge of the inscribed angle is a diameter of the circle. Inscribed angle is α. Central angle is β.



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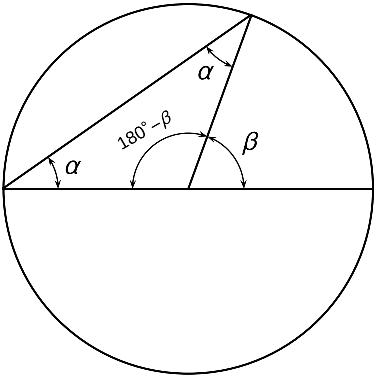

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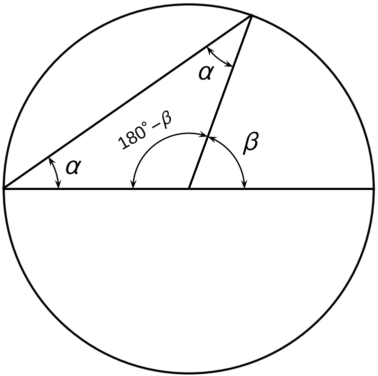

Since the angles of a triangle must sum to 180°, then

$$180° - \beta + \alpha + \alpha = 180° \implies \boxed{\beta = 2\alpha}$$

It is easy to generalize this argument to any inscribed angle and its corresponding central angle. . .

Circles: Central and Inscribed Angles: Diameter Included

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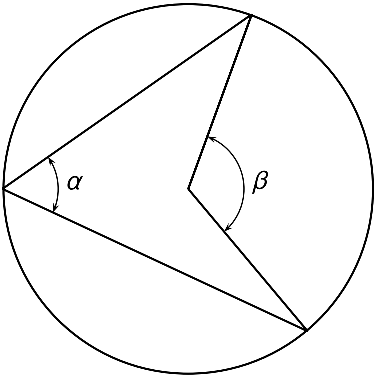
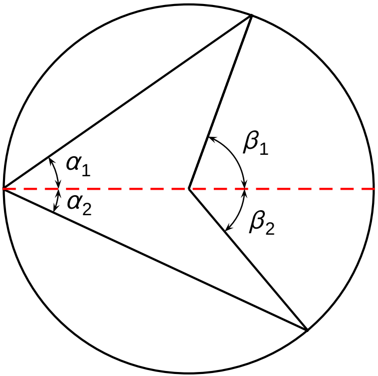


Circles: Central and Inscribed Angles: Diameter Included

Now we consider the case where the inscribed angle includes a diameter of the circle.
Add the diameter and consider the two pairs of angles (α_1, β_1) and (α_2, β_2), where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$. By our previous result, $\beta_1 = 2\alpha_1$ and $\beta_2 = 2\alpha_2$, so that



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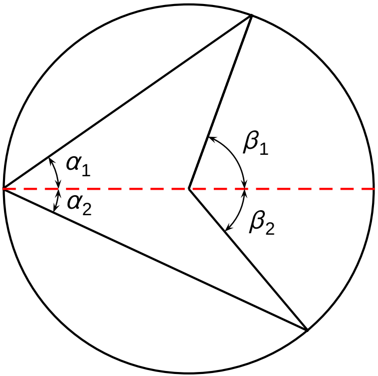

$$\beta = \beta_1 + \beta_2 = 2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2) = 2\alpha$$

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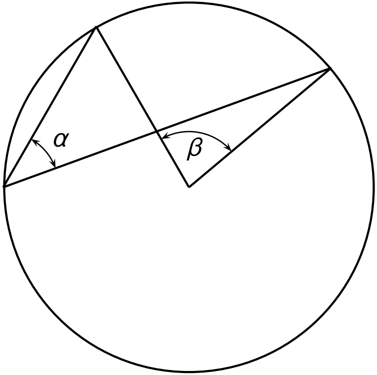
Now we consider the case where the inscribed angle excludes the diameter of the circle.



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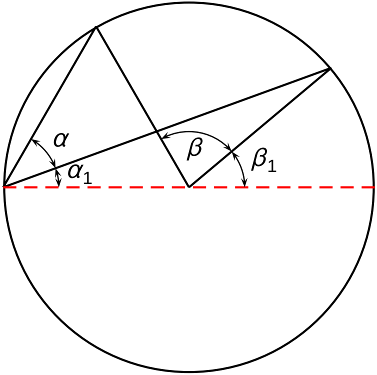

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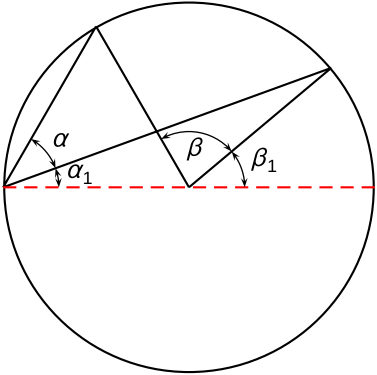

$$\beta = (\beta + \beta_1) - \beta_1 = 2(\alpha + \alpha_1) - 2\alpha_1 = 2\alpha$$

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One could solve the two equations for the two unknowns x and y, but it is simpler to use the identity

$$x^2 - y^2 = (x + y)(x - y) = \frac{7}{12} \times \frac{1}{12} = \boxed{\frac{7}{144}}$$

Problem 10

When simplified, what is the value of
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Since we are multiplying factors with common bases, we add exponents:

$$10^{0.5+0.3+0.2+0.1+0.9} = 10^2 = \boxed{100}$$