Warm-Up 7 Solutions

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November 3, 2004

Problem 1

Jane spent the following amounts on her school lunches last Monday through Thursday: \$3.29 - Mon., \$1.79 - Tues., \$2.65 - Wed., and \$3.29 - Thurs. After she bought her lunch on Friday, she calculated that her average lunch cost for that week was \$2.80 per day. How much did her lunch cost on Friday?

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The total cost of lunch for the week was $2.80 \times 5 = \$14$. So she must have spent

$$14 - (3.29 + 1.79 + 2.65 + 3.29) = 14 - 11.02 = \boxed{\$2.98}$$

on lunch Friday.

Problem 2

Four straight lines intersect a circular region. The lines and circle are coplanar, and two of the lines are parallel. What is the maximum number of non-overlapping regions inside the circle?

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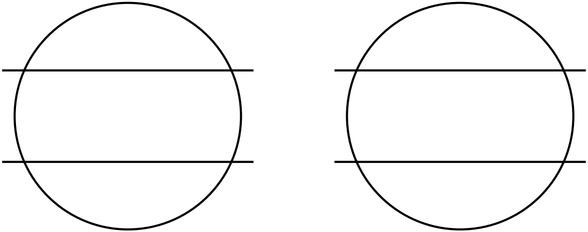
We might as well draw the parallel lines first:



To maximize the number of regions in the circle, the other two lines should intersect each other and the two parallel lines within the circle.

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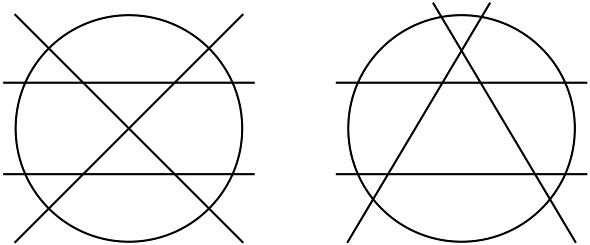

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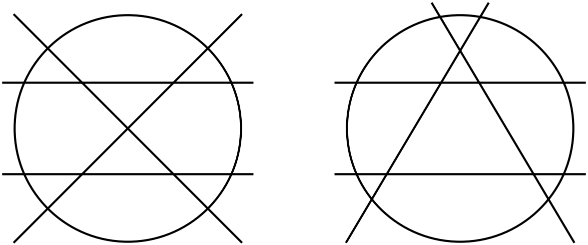

To maximize the number of regions in the circle, the other two lines should intersect each other and the two parallel lines within the circle. The maximum number of regions is $\boxed{10}$.

Problem 3

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Let's check a few powers of 7:

$$7^2 = 49, \quad 7^3 = 343, \quad 7^4 = 2401, \quad 7^5 = 16807, \quad 7^6 = 117649,$$
$$7^7 = 823543, \quad 7^8 = 5764801, \quad 7^9 = 40353607, \ldots$$

Can you see the pattern?

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Can you see the pattern? If the remainder (when the exponent is divided by 4) is 0 or 1, the tens digit is 0. If the remainder is 2 or 3, the tens digit is 4:

Exponent n	10's digit of 7^n	Remainder of $n/4$
2	4	2
3	4	3
4	0	0
5	0	1
6	4	2
7	4	3

Problem 3, Continued

So, the remainder when 2005 is divided by 4 is 1, and the tens digit of 7^{2005} is $\boxed{0}$.

Problem 4

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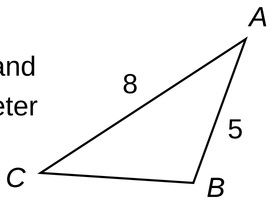
Since the product of the digits is zero, then at least one digit is 0. Since it is a three-digit number, the most significant digit is not zero. Since it is an odd number, the least significant digit (LSD) is not zero. Therefore, the middle digit is zero. Since n is odd, the LSD is odd. So the only possibilities are 107, 701, 305, and 503. Their mean is

$$(107 + 701 + 305 + 503)/4 = \boxed{404}$$

Problem 5

Triangle ABC is isosceles with vertex angle B and legs $\overline{BC}$ and $\overline{AB}$, as shown. What is the perimeter of triangle ABC?



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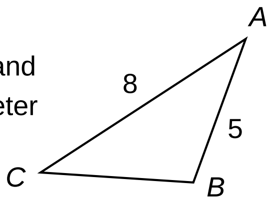
Triangle ABC is isosceles with vertex angle B and legs $\overline{BC}$ and $\overline{AB}$, as shown. What is the perimeter of triangle ABC?



Since the legs of an isosceles triangle are congruent, we know that $BC = AB = 5$, and the perimeter is

$$\text{Perimeter} = 8 + 5 + 5 = \boxed{18}$$

Problem 6

In a school of 250 students, everyone takes one English class and one history class each year. Today, 15 total students were absent from their English class and ten total students were absent from their history class. Five of the students were absent from both classes. If a student is chosen at random from this school, what is the probability that s/he was not absent from either class? Express your answer as a percent.

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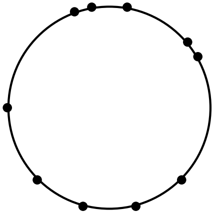
5 students missed English only, 10 missed History only, and 5 missed both History and English. Therefore 20 students missed one or both classes, and 250 – 20 = 230 students were not absent from either class. This represents

$$\frac{230}{250} = \frac{23}{25} = \frac{92}{100} = \boxed{92\%}$$

of all the students.

Problem 7

Ten distinct points are identified on the circumference of a circle. How many different convex quadrilaterals can be formed if each vertex must be one of these 10 points?



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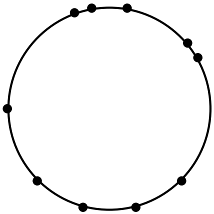

Definition

A polygon is said to be convex if it fully contains all line segments drawn between any two points of the polygon.

Once four vertices have been selected, the quadrilateral is determined, regardless of the order of the vertices. So this is just a combination problem. We seek the number of combinations of 10 things taken 4 at a time:

$$C(10, 4) = {}_{10}C_4 = \frac{10!}{4!\,(10-4)!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 10 \times 3 \times 7 = \boxed{210}$$

Problem 8

Thad has an unlimited supply of 3-cent and 4-cent stamps. If he has to put exactly 37 cents of postage on a letter, how many different combinations of 3-cent and/or 4-cent stamps could Thad use?

Problem 8, Continued

If x is the number of 3-cent stamps and y is the number of 4-cent stamps, then $3x + 4y = 37$, where x and y are both non-negative integers. The maximum that x can be is 12 (why?), so we can just check that each $y = (37 - 3x)/4$ is a nonnegative integer:

x	$y = (37 - 3x)/4$
1	8.50
2	7.75
3	7
4	6.25
5	5.50
6	4.75
7	4
8	3.25
9	2.50
10	1.75
11	1
12	0.25

So there are $\boxed{3}$ such combinations of stamps.

Problem 9

On a graph, a lattice point is an ordered pair (x, y) with integers x and y. Exactly 15 lattice points lie strictly in the interior of the triangular region with vertices $(0, 0)$, $(N, 0)$ and (N, N), where $N > 0$. What is the value of N?

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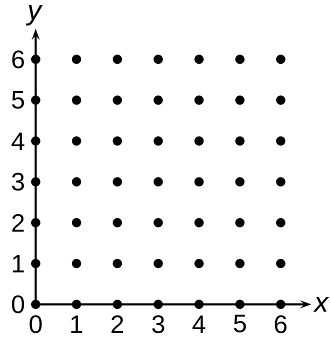
Let's try out some small values of N and see how many points are inside the triangle:

N	# Points Inside Triangle



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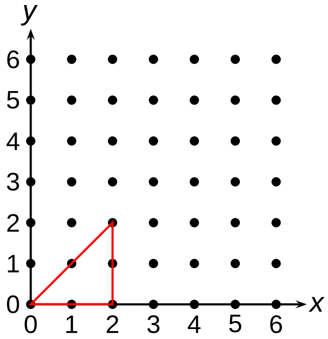

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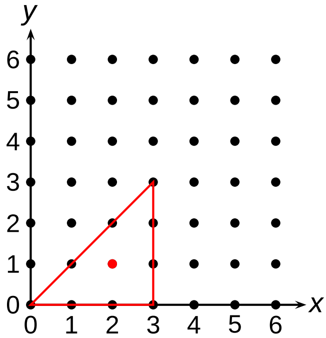
N	# Points Inside Triangle
2	0
3	1



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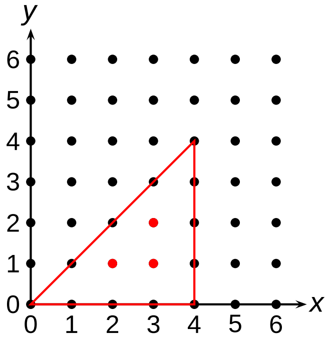

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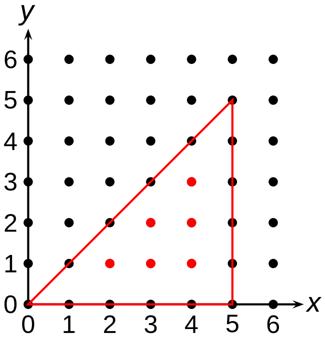
5	$1 + 2 + 3 = 6 = T_3$



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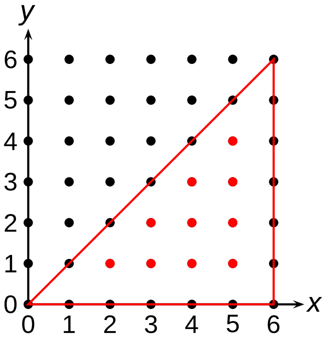
2	0
3	1
4	$1 + 2 = 3 = T_2$
5	$1 + 2 + 3 = 6 = T_3$
6	$1 + 2 + 3 + 4 = 10 = T_4$



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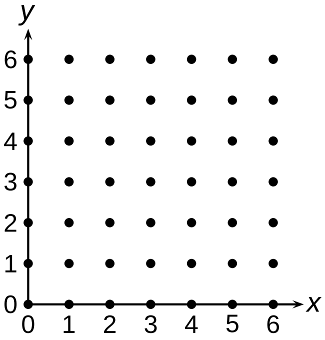
3	1
4	$1 + 2 = 3 = T_2$
5	$1 + 2 + 3 = 6 = T_3$
6	$1 + 2 + 3 + 4 = 10 = T_4$
$\vdots$	$\vdots$
n	$1 + 2 + \cdots + (n-2) = T_{n-2}$
$\vdots$	$\vdots$



Problem 9, Continued

So we have to find the number N such that

$$T_{N-2} = 15$$

But we remember that

$$T_5 = 1 + 2 + \cdots + 6 = \frac{5 \times 6}{2} = 15$$

so that

$$N - 2 = 5 \implies N = \boxed{7}$$

Problem 10

According to the linear function represented in this table, what is the value of x when $y = 8$?

x	y
-4	23
1	20
6	17

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Note that the x values in the table are changing by $+5$ while the y values are changing by -3. If we fill in the next few entries in the table, we find the correct x value is

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According to the linear function represented in this table, what is the value of x when $y = 8$?

x	y
-4	23
1	20
6	17
11	14
16	11

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Problem 10

According to the linear function represented in this table, what is the value of x when $y = 8$?

x	y
-4	23
1	20
6	17
11	14
16	11
21	8

Note that the x values in the table are changing by $+5$ while the y values are changing by -3. If we fill in the next few entries in the table, we find the correct x value is $\boxed{21}$.