Warm-Up 4 Solutions

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Problem 1

Today Darron's teacher pairs each student with a partner to create exactly 12 pairs of students. Next week each student will be paired with a different partner. Darron's partner for next week can only be chosen from how many students?

With exactly 12 pairs, there must be 24 students in the class. Since Darron cannot be paired with himself nor with his first partner, that leaves $\boxed{22}$ possibilities.

Problem 2

Zan has created this rule for generating sequences of whole numbers:

If a number is 15 or less, triple the number.

If a number is more than 15, subtract 13 from it.

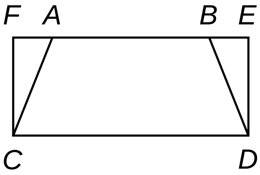
Therefore, if Zan starts with 10, she gets the sequence $10, 30, 17, 4, 12, \ldots$. If the first number in Zan's sequence is 34, what is the 8th number in the sequence?

The sequence is

$$34 \to 21 \to 8 \to 24 \to 11 \to 33 \to 20 \to \boxed{7}$$

Problem 3

Isosceles trapezoid *ABDC* and rectangle *CDEF* are shown, with $DE = 5\,\text{in}$ and $CD = 12\,\text{in}$. The area of trapezoid *ABDC* is $50\,\text{in}^2$. What is *BE*?



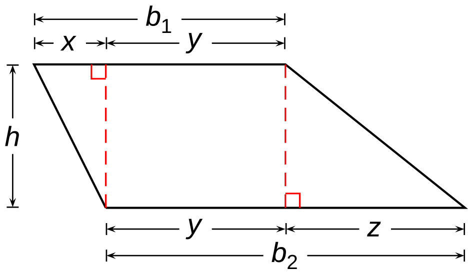
Definition

A trapezoid is a quadrilateral (four-sided plane figure) with exactly one pair of opposite sides (called the *bases* parallel.

In an equilateral trapezoid the nonparallel sides are congruent, and they make equal angles with the base.

Problem 3: The Area of a Trapezoid

We find the area of a trapezoid by decomposing the trapezoid into right triangles and a rectangle:

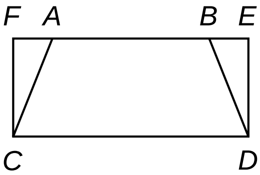


$$\text{Trapezoid area} = \frac{1}{2}xh + yh + \frac{1}{2}zh = \left(\frac{1}{2}xh + \frac{1}{2}yh\right) + \left(\frac{1}{2}yh + \frac{1}{2}zh\right)$$

$$= \frac{1}{2}(x+y)h + \frac{1}{2}(y+z)h = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \boxed{\frac{b_1+b_2}{2}h}$$

Problem 3 Solution

Isosceles trapezoid $ABDC$ and rectangle $CDEF$ are shown, with $DE = 5$ in and $CD = 12$ in. The area of trapezoid $ABDC$ is $50\,\text{in}^2$. What is BE?



$$\text{Area} = 50 = \frac{AB + CD}{2} DE = \frac{AB + 12}{2} \times 5$$

$$AB + 12 = \frac{2}{5} \times 50 = 20 \implies AB = 8$$

We can now find BE as

$$BE = FA = \frac{1}{2}(CD - AB) = \frac{1}{2}(12 - 8) = \boxed{2}$$

Problem 4

How many positive factors does 48 have?

We first write 48 in terms of its prime factors:

$$48 = 2 \times 24 = 2^2 \times 12 = 2^3 \times 6 = 2^4 \times 3^1$$

Now any factor f of 48 must be of the form

$$f = 2^m \times 3^n, \qquad \text{where } m \in \{0, 1, 2, 3, 4\}, \quad n \in \{0, 1\}$$

Since there are 5 choices for m and independently there are 2 choices for n, the fundamental principle of counting tells us that there are

$$5 \times 2 = \boxed{10}$$

choices for selecting the pair (m, n) and thus $\boxed{10}$ factors.

Problem 5

Two quantities a and b are said to vary inversely if the value of the product ab remains constant. The number of questions q on a test varies inversely with the number of points p that each question is worth. If $q = 20$ when $p = 5$, what is the value of q when $p = 2$?

We are told that pq is always the same value. For $p = 5$ and $q = 20$ we have $pq = 100$. So when $p = 2$ we must have

$$q = \frac{pq}{p} = \frac{100}{2} = \boxed{50}$$

Question

How many points are on the test?

Problem 6

How many digits does the smallest repeating block in the decimal expansion of $\frac{5}{7}$ contain?

Using a calculator or by long division we find that

$$\frac{5}{7} = 0.71428571428571\ldots = 0.\overline{714285}$$

so that the repeating block consists of the $\boxed{6}$ digits 714285.

Problem 7

A pair of six-sided dice is rolled, and the sum is recorded. What is the probability that this sum is a multiple of three? Express your answer as a common fraction.

Each roll will result in a number from 1 to 6. There are $6 \times 6 = 36$ possible outcomes from rolling a pair of dice by the fundamental principle of counting. Let's make a table listing possible favorable outcomes:

1st Roll	Possible 2cd Rolls	# Ways
1	2, 5	2
2	1, 4	2
3	3, 6	2
4	2, 5	2
5	1, 4	2
6	3, 6	2
	Total	12

Problem 7, Continued

So there are 12 ways to roll a pair of dice whose sum is a multiple of 3. Since there are 36 possible outcomes when rolling a pair of dice, then the probability of getting such a sum is

$$\text{Probability} = \frac{12}{36} = \boxed{\frac{1}{3}}$$

Problem 8

The chart below was created from the results of a radio station survey. What percent of the males surveyed listen to the station?

	Listen to Station		Total
	Yes	No	
Male	39%	13%	52%
Female	29%	19%	48%
Total	68%	32%	100%

Note that 39% is the fraction of people surveyed who are both male and listen to the station. 13% is the fraction of people surveyed who are both male and do not listen to the station. So the percentage of all males surveyed that listen to the station is

$$\frac{39}{39+13} \times 100\% = \frac{39}{52} \times 100\% = \boxed{75\%}$$

Problem 9

What is the value of $1 + 4\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 1\left(\frac{1}{2}\right)^4$? Express your answer as a common fraction.

Note that the coefficients appearing above $(1, 4, 6, 4, 1)$ are those that occur in the expansion of a binomial raised to the fourth power:

$$(a+b)^4 = 1a^0b^4 + 4a^1b^3 + 6a^2b^2 + 4a^3b^1 + 1a^4b^0.$$

Choosing $a = \frac{1}{2}$ and $b = 1$ yields

$$\begin{aligned}
\left(\frac{1}{2}+1\right)^4 &= 1\left(\frac{1}{2}\right)^0 1^4 + 4\left(\frac{1}{2}\right)^1 1^3 + 6\left(\frac{1}{2}\right)^2 1^2 + 4\left(\frac{1}{2}\right)^3 1^1 + 1\left(\frac{1}{2}\right)^4 1^0 \\
&= 1 + 4\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 1\left(\frac{1}{2}\right)^4 \\
&= \left(\frac{3}{2}\right)^4 = \left(\frac{9}{4}\right)^2 = \boxed{\frac{81}{16}}
\end{aligned}$$

Problem 10

A bug crawls at a rate of 14π units per hour around a circle with a radius of 3.5 units. How many hours does it take the bug to complete 26 revolutions of the circle?

The circumference C of the circle is given by the formula

$$C = 2\pi r = 2\pi \times 3.5 = 7\pi.$$

We know that Distance = Speed × Time or

$$\begin{aligned} \text{Time} &= \frac{1}{\text{Speed}} \times \text{Distance} \\ &= \frac{1\,\text{hr}}{14\pi\,\text{units}} \times (26 \times 7\pi\,\text{units}) \\ &= \boxed{13\,\text{hr}} \end{aligned}$$