Warm-Up 16 Solutions

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Homework: January 26, 2005

Problem 1

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The midpoint M is $7 - (-2) = 7 + 2 = 9$ units to the right of A. Point B must be 9 units to the right of M, so

$$B = 7 + 9 = \boxed{16}$$

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The prime numbers less than 50 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

- All two-digit prime numbers are odd.
- The difference between an even number and an odd number is odd.
- The difference between two odd numbers is even.

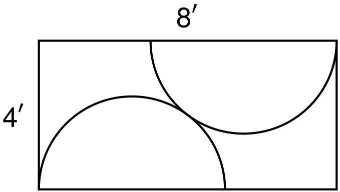
We conclude that the primes we seek must begin with an odd digit in the tens place. These numbers are:

$$11, 13, 17, 19, 31, 37$$

of which there are $\boxed{6}$.

Problem 3

A carpenter wants to make the largest possible circular tabletop from a $4' \times 8'$ sheet of plywood. If the tabletop is constructed from the two largest congruent semi-circular pieces that can be cut from the sheet, what is the diameter of the resulting table? (Assume no wood is lost when cutting the plywood.)



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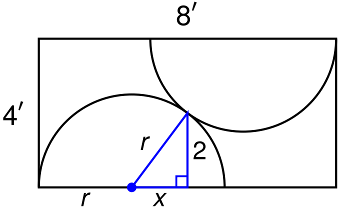

$$x = \sqrt{r^2 - 4}$$

By symmetry, the point of tangency of the two circles is at the center of the rectangle. Therefore, the height of the blue triangle is 2, and

$$4 = r + x = r + \sqrt{r^2 - 4} \implies \sqrt{r^2 - 4} = 4 - r$$

$$r^2 - 4 = (4 - r)^2 = 16 - 8r + r^2$$

$$8r = 16 + 4 = 20$$

$$\text{Diameter} = 2r = \frac{20}{4} = \boxed{5}$$

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$$3f(m) = 3(m^2 + 12) = 3m^2 + 36$$
$$f(3m) = (3m)^2 + 12 = 3^2m^2 + 12 = 9m^2 + 12$$

Equating these two, we obtain an equation that can be solved for m:

$$3m^2 + 36 = 9m^2 + 12$$
$$\Longrightarrow\ 0 = 6m^2 - 24 = 6(m^2 - 4) = 6(m-2)(m+2)$$

The roots are $m = \pm 2$ and since $m > 0$, we conclude $m = \boxed{2}$.

Problem 5

A piece of wire 180 cm long is cut into two pieces with integer lengths. Each of the two pieces is formed into its own square with integer side lengths. The total area of the two squares is 1073 cm^2. How many centimeters longer is a side of the larger square than a side of the smaller square?

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Let the side lengths of the squares be m and n. We are given the following two equations:

$$4m + 4n = 180 \implies m + n = 45$$
$$m^2 + n^2 = 1073$$

From the first equation we have that $n = 45 - m$, which can be substituted into the second equation:

$$1073 = m^2 + (45 - m)^2 = m^2 + 2025 - 90m + m^2$$
$$2m^2 - 90m + 952 = 0 \implies m^2 - 45m + 476 = (m - 28)(m - 17) = 0$$

So the two side lengths are 28 and 17. Their difference is $\boxed{11}$

Problem 6

Calculate the sum of the geometric series $1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$. Express your answer as a common fraction.

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Calculate the sum of the geometric series $1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$. Express your answer as a common fraction.

Let $S = 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$. Then

$$\begin{aligned} 3S &= 3\left[1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \cdots\right] \\ &= 3 + 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots \\ &= 3 + S \end{aligned}$$

So that

$$3S = 3 + S \implies 2S = 3 \implies S = \boxed{\frac{3}{2}}$$

Problem 7

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Suppose $x = 10a + b$ and $y = 10b + a$, where a and b are distinct decimal digits in the written representation of x and y. Then

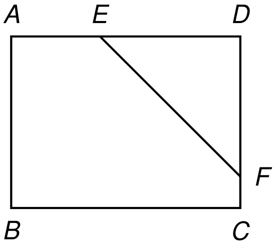
$$x + y = 10a + b + 10b + a = 11a + 11b = 11(a + b)$$

so that each sum $x + y$ must be a multiple of 11. The least such sum is $12 + 21 = 33 = 3 \times 11$ and the greatest is $98 + 89 = 187 = 17 \times 11$. Since all intervening multiples of 11 can also occur, the number of such sums is

$$17 - 3 + 1 = \boxed{15}$$

Problem 8

In rectangle $ABCD$, $AB = 6\,\text{cm}$, $BC = 8\,\text{cm}$, and $DE = DF$. The area of triangle DEF is one-fourth the area of rectangle $ABCD$. What is the length of segment $\overline{EF}$? Express your answer in simplest radical form.



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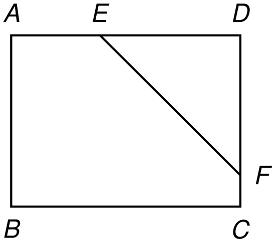

The area of the triangle is $\frac{1}{2}DE^2$ and the area of the rectangle is $6 \times 8 = 48$. We are told that

$$\frac{1}{2}DE^2 = \frac{1}{4} \times 48 = 12 \implies DE^2 = 24 \implies DE = \sqrt{24} = 2\sqrt{6}$$

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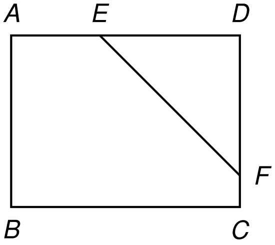

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Since DEF is a right, isosceles triangle, the length of its hypotenuse, $\overline{EF}$, is related to its legs via

$$EF = \sqrt{2}DE = \sqrt{2} \times \sqrt{6} = \sqrt{2} \times \sqrt{3 \times 2} = \boxed{2\sqrt{3}}$$

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Suppose that packages 1 and 2 go to the correct houses. Then there is only one way for the other two packages to be delivered so that exactly 2 are correct: they must be swapped, so that the packages are ordered $(1, 2, 4, 3)$. It's the same for any other choice of 2 packages correctly delivered—the remaining 2 must always be swapped. So the number of ways to deliver exactly 2 packages correctly is the number of ways to choose 2 from 4, or

$${}_4C_2 = \frac{{}_4P_2}{{}_2P_2} = \frac{4 \times 3}{2 \times 1} = \frac{4!}{2!\,2!} = 6$$

Problem 9, Continued

so the probability of exactly 2 packages being delivered correctly is

$$\frac{6}{24} = \boxed{\frac{1}{4}}$$

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Using the sides of the 64 unit squares of a standard chessboard, how many non-congruent rectangles can be formed? Two such rectangles (1×4 and 4×3) are identified to the right.

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There are 8 ways to form rectangles where the first dimension is 1:

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There are 7 (new) ways to form rectangles where the first dimension is 2:

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There are 7 (new) ways to form rectangles where the first dimension is 2:

$$2 \times 2,\ 2 \times 3,\ 2 \times 4,\ 2 \times 5,\ 2 \times 6,\ 2 \times 7,\ 2 \times 8$$

There are 6 (new) ways to form rectangles where the first dimension is 3:

$$3 \times 3,\ 3 \times 4,\ 3 \times 5,\ 3 \times 6,\ 3 \times 7,\ 3 \times 8$$

Problem 10, Continued

Continuing this way, we find that each time we increase the first dimension, we reduce by 1 the number of new ways to create rectangles, until finally, there is only a single (new) way to make a rectangle with 8 as the first dimension: 8×8. So the total number of ways to make rectangles is

$$1 + 2 + \cdots + 8 = T_8 = \frac{8 \times 9}{2} = 4 \times 9 = \boxed{36}$$