Warm-Up 15 Solutions

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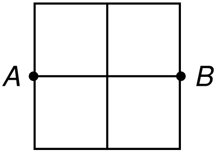
Quiz: January 26, 2005

Problem 1

Raquel colors in this figure so that each of the four unit squares is completely red or completely green. In how many different ways can the picture be colored this way so that there is a horizontal line of symmetry at $\overline{AB}$?



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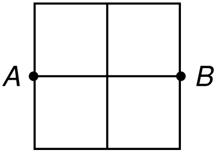

If we insist that $\overline{AB}$ is a line of symmetry, then the bottom squares must be colored the same as the corresponding top squares, so that there are only 2 squares that can be independently colored: the top left and top right squares. Since there are two choices for each of these squares, then there are $2 \times 2 = \boxed{4}$ ways to color the squares: RR, RG, GR, GG.

Problem 2

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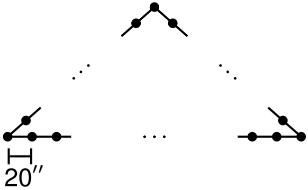
The even, two-digit multiples of 11 are 22, 44, 66, and 88. Since $8 = 2^3$, then

$$8 \times 8 = \left(2^3\right)^2 = 2^{3\times 2} = \left(2^2\right)^3 = 4^3$$

so the answer is $\boxed{88}$.

Problem 3

A fence encloses a triangular region with sides measuring 30 feet, 20 feet and 20 feet. The fence posts are placed 20 inches apart, as shown. How many fence posts are needed to enclose the triangular region with the fence?



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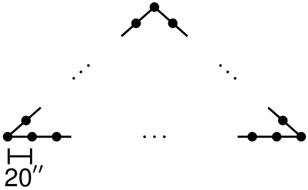

The posts divide the perimeter into a number of segments. For a closed perimeter, the number of posts will be equal to the number of segments. Since each segment is 20 inches long, and the perimeter of the figure is $30 + 20 + 20 = 70$ feet or $70 \times 12 = 840$ inches, the number of segments is

$$\# \text{ Segments} = \# \text{ Posts} = \frac{840}{20} = \boxed{42}$$

Problem 4

Bryce bought 32 stools that required assembly from Need-A-Seat. Some stools have three legs, and the other stools have four legs. The box arrived with 108 stool legs. If the four-legged stools cost $20 and the three-legged stools cost $15, how much did all of Bryce's stools cost?

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The cost per leg of the three-legged stools is $15/3$ = \$5 per leg. The cost per leg for the four-legged stool is $20/4$ = \$5 per leg. So the total cost of 108 stool legs is

$$\$5 \times 108 = \boxed{\$540}$$

Problem 5

One pump can empty a tank in eight hours. A second pump can empty the same tank in five hours. What is the positive difference between the time it would take the faster pump to empty the tank working alone and the time it would take for the two pumps to empty the tank working together? Express your answer as a common fraction.

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If the volume of the tank is T gallons, then the first pump empties $T/8$ gallons per hour and the second pump empties $T/5$ gallons per hour. Operating together, the pumps can remove

$$\frac{T}{5} + \frac{T}{8} = \frac{5T + 8T}{40} = \frac{13T}{40} \text{ gal/hr}$$

The time needed for the two pumps to empty the tank is

$$T \text{ gal} \times \frac{40}{13T} \text{ hr/gal} = \frac{40}{13} \text{ hr}$$

and the positive difference requested above is

$$5 - \frac{40}{13} = \frac{65 - 40}{13} = \boxed{\frac{25}{13} \text{ hr}}$$

Problem 6

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Factoring the polynomial, we get

$$P(x) = (2x-1)(2x+1)^2 = 8\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)^2$$

The factor $(x - 1/2)$ is negative for $x < \frac{1}{2}$ and positive for $x > \frac{1}{2}$. Since the other factor is squared, it is always non-negative, and is positive for $x > -\frac{1}{2}$. So the entire expression will be positive for $x > \frac{1}{2}$, thus

$$a = \boxed{\frac{1}{2}}$$

Problem 7

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Let the original collection of integers be $\{a, b, c, 3, d, e, f)$, where

$$1 \leq a \leq b \leq c \leq 3 \leq d \leq e \leq f$$

Since one of the numbers less than or equal to 3 must be repeated, and the unique mode is 4, then three of the entries must be 4:

$$(a, b, c, 3, 4, 4, 4)$$

If adding a pair of 2s results in 2 being the unique mode, then exactly two of the original entries must also have been 2. The original set is then either

$$(1, 2, 2, 3, 4, 4, 4) \text{ or } (2, 2, 3, 3, 4, 4, 4)$$

Problem 7, Continued

After augmenting these with a pair of 2s, the new sets become

$$(1, 2, 2, 2, 2, 3, 4, 4, 4) \text{ or } (2, 2, 2, 2, 3, 3, 4, 4, 4)$$

Only the first set above has a median value of 2. Its mean value is

$$\frac{1 + 2 + 2 + 2 + 2 + 3 + 4 + 4 + 4}{9} = \boxed{\frac{24}{9}}$$

Problem 8

A box contains two coins with a Head on both sides, one standard coin and one coin with a Tail on both sides. A coin will be randomly selected from these four coins and will be flipped twice. What is the probability that each of the two flips will result in a Head? Express your answer as a common fraction.

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There are two ways to obtain the two heads. One can draw the two-headed coin or the standard coin. The probability of drawing the two-headed coin is $1/2$, in which case obtaining two heads is certain. The probability of drawing the standard coin is $1/4$. If the standard coin is drawn, the *conditional probability* of obtaining two heads is $1/4$.

So the probability of obtaining two consecutive heads is

$$\frac{1}{2} + \frac{1}{4} \times \frac{1}{4} = \frac{8}{16} + \frac{1}{16} = \boxed{\frac{9}{16}}$$

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Suppose a set has n elements. When forming subsets, for each element of the original set, we can include it in the subset or not. So there are two choices for each element of the original set, and the fundamental principle of counting tells us that the number of subsets we can make is $\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n \text{ factors}} = 2^n$.

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Let the number of elements in sets A and B be n_A and n_B, respectively. We are told that $n_A = n_B + 2$ and $2^{n_A} = 96 + 2^{n_B}$ or

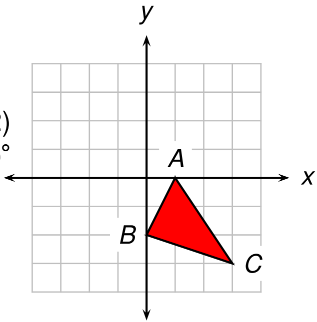
$$\begin{aligned} 96 &= 2^{n_A} - 2^{n_B} = 2^{n_A} - 2^{n_A - 2} = 2^{n_A} - 2^{n_A} \cdot 2^{-2} \\ &= 2^{n_A}(1 - 2^{-2}) = 2^{n_A}\left(1 - \frac{1}{4}\right) = 2^{n_A}\frac{3}{4} \end{aligned}$$

$$2^{n_A} = \frac{4}{3} \cdot 96 = 128$$

so that $n_A = \boxed{7}$.

Problem 10

Triangle ABC has vertices $A(1, 0)$, $B(0, -2)$ and $C(3, -3)$. If triangle ABC is rotated 90° counterclockwise about A, what are the coordinates of the image of C?



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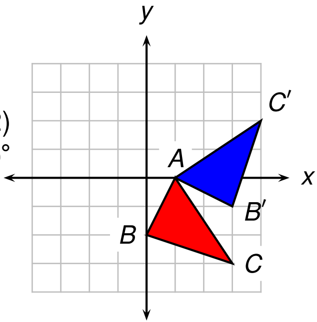

Before rotation, point C is located 2 units to the right of and 3 units below point A. After rotation, point A will not have moved, while point C' will be located on a line at right angles to $\overline{AC}$. It will be 3 units to the right and 2 units above point A. Its coordinates will be

$$(1, 0) + (3, 2) = \boxed{(4, 2)}$$