Warm-Up 14 Solutions

Peter S. Simon

January 12, 2005

Problem 1

Ten cards are numbered and lying face up in a row, as shown. David turns over every card that is a multiple of 2. Then he turns over every card that is a multiple of 3, even if the card had been turned over previously and is currently face down. He continues this process with the multiples of 4 through 9. How many cards are then face up?

$\boxed{0}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$ $\boxed{10}$

Problem 1, Continued

Card Value	1	2	3	4	5	6	7	8	9	10
Mult of 2?	0	1	0	1	0	1	0	1	0	1
Mult of 3?	0	0	1	0	0	1	0	0	1	0
Mult of 4?	0	0	0	1	0	0	0	1	0	0
Mult of 5?	0	0	0	0	1	0	0	0	0	1
Mult of 6?	0	0	0	0	0	1	0	0	0	0
Mult of 7?	0	0	0	0	0	0	1	0	0	0
Mult of 8?	0	0	0	0	0	0	0	1	0	0
Mult of 9?	0	0	0	0	0	0	0	0	1	0
Total # Flips	0	1	1	2	1	3	1	3	2	2

The cards with an even number of flips end up face-up. These include the $\boxed{1}$, $\boxed{4}$, $\boxed{9}$, and $\boxed{10}$. There are $\boxed{4}$ such cards.

Problem 2

The equation $x^2 + bx + 36 = 0$ has two distinct negative, integer solutions. What is the sum of all of the distinct possible integer values for b?

Problem 2

The equation $x^2 + bx + 36 = 0$ has two distinct negative, integer solutions. What is the sum of all of the distinct possible integer values for b?

If the equation has two integer solutions m and n, then it can be factored into the form

$$0 = (x - m)(x - n) = x^2 - (m + n)x + mn = x^2 + bx + 36$$

so $mn = 36$ and $b = -(m + n)$, with m and n being distinct, negative integers. We list the possibilities below:

m	n	$b = -(m + n)$
-1	-36	37
-2	-18	20
-3	-12	15
-4	-9	13
Sum of b values		$\boxed{85}$

Problem 3

How many integers between 100 and 500 have at least two 3s as digits?

Problem 3

How many integers between 100 and 500 have at least two 3s as digits?

In the three hundreds, the qualifying numbers are 330–339, also 303, 313, 323, 343, 353, 363, 373, 383, and 393, for a total of 19 numbers. The other numbers are 133, 233, and 433, so the total number is $19 + 3 = \boxed{22}$.

Problem 4

The average of two 2-digit positive integers is equal to the decimal number obtained by writing one of the two-digit integers before the decimal point and the other two-digit integer after the decimal point. What is the smaller of the two integers?

Problem 4

The average of two 2-digit positive integers is equal to the decimal number obtained by writing one of the two-digit integers before the decimal point and the other two-digit integer after the decimal point. What is the smaller of the two integers?

Let the integers be a and b. The average is $(a+b)/2$, and the number obtained by writing $a.b$ is numerically equal to $a + \frac{b}{10}$. We are told these are equal:

$$\frac{a+b}{2} = a + \frac{b}{10} \implies a - \frac{a}{2} = \frac{b}{2} - \frac{b}{10} \implies \frac{a}{2} = \frac{4b}{10} \implies a = \frac{4}{5}b$$

so a is the smaller of the two digits. The obvious choice is $a = \boxed{4}$ and $b = 5$, which works, since

$$\frac{4+5}{2} = \frac{9}{2} = 4.5$$

Problem 5

What is the value of $\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\cdots}}}}$?

Problem 5

What is the value of $\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\cdots}}}}$?

What does an expression such as the one above mean? How do we assign a value to it? Consider the following sequence:

$$\sqrt{20} \approx 4.4721$$

$$\sqrt{20+\sqrt{20}} \approx 4.9469$$

$$\sqrt{20+\sqrt{20+\sqrt{20}}} \approx 4.9947$$

$$\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20}}}} \approx 4.9995$$

$$\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20}}}}} \approx 4.9999$$

$$\vdots$$

Problem 5, Continued

The numbers appear to be getting closer and closer to some number that we call the limit of the sequence. Let us call this limit x. Then

$$x = \sqrt{20 + \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}}$$

which we can write as

$$x = \sqrt{20 + x} \implies x^2 = 20 + x \implies x^2 - x - 20 = 0 \implies (x - 5)(x + 4) = 0$$

The two possible solutions are $x = 5$ and $x = -4$. Since x is clearly positive, we accept the solution $\boxed{5}$.

Problem 6

There are equal numbers of pennies, nickels, dimes and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins will be less than 20 cents? Express your answer as a common fraction.

Problem 6

There are equal numbers of pennies, nickels, dimes and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins will be less than 20 cents? Express your answer as a common fraction.

- Because there are equal numbers of each type of coin, the probability of drawing any one denomination on a single draw is $\frac{1}{4}$.

Problem 6

There are equal numbers of pennies, nickels, dimes and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins will be less than 20 cents? Express your answer as a common fraction.

- Because there are equal numbers of each type of coin, the probability of drawing any one denomination on a single draw is $\frac{1}{4}$.
- Because the coins are replaced each time, the probability remains the same from draw to draw.

Problem 6

There are equal numbers of pennies, nickels, dimes and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins will be less than 20 cents? Express your answer as a common fraction.

- Because there are equal numbers of each type of coin, the probability of drawing any one denomination on a single draw is $\frac{1}{4}$.
- Because the coins are replaced each time, the probability remains the same from draw to draw.
- There are $4 \times 4 \times 4 \times 4 = 256$ possible outcomes of drawing the four coins, so each particular outcome has probability $\frac{1}{256}$.

Problem 6

There are equal numbers of pennies, nickels, dimes and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins will be less than 20 cents? Express your answer as a common fraction.

- Because there are equal numbers of each type of coin, the probability of drawing any one denomination on a single draw is $\frac{1}{4}$.
- Because the coins are replaced each time, the probability remains the same from draw to draw.
- There are $4 \times 4 \times 4 \times 4 = 256$ possible outcomes of drawing the four coins, so each particular outcome has probability $\frac{1}{256}$.
- The combinations of four coins that result in a value less than 20 cents are $\{P, P, P, P\}$, $\{N, P, P, P\}$, $\{N, N, P, P\}$, $\{N, N, N, P\}$, $\{N, D, P, P\}$, $\{D, P, P, P\}$.

Problem 6 Side Note: Permutations with Repetitions

Permutations of "DADDY" Suppose we want to find the number of five-letter words that can be created by rearranging the letters in the word "DADDY". First, consider the number of permutations of the symbols $D_1AD_2D_3Y$, where we will temporarily distinguish between the three D's. Clearly there are ${}_5P_5 = 5!$ such permutations. However, we note that the following six permutations $D_1D_2D_3AY$, $D_1D_3D_2AY$, $D_2D_1D_3AY$, $D_2D_3D_1AY$, $D_3D_1D_2AY$, and $D_3D_2D_1AY$ all produce the same word when the subscripts are removed. The 6 comes from the fact that there are ${}_3P_3 = 3! = 6$ permutations for rearranging the three D's within the first three positions of this permutation. This will be true for any choice of placement of the three D's. Thus there are

$$\frac{5!}{3!} = \frac{120}{6} = 20$$

different five-letter words obtainable by rearranging the word DADDY.

Permutations of “DADDA”

How many five-letter words can be formed by rearranging the letters in the word “DADDA”? As before, we note that if all the letters were distinguishable, then we would have ${}_5P_5 = 5!$ possible rearrangements. However, three D’s are identical and two A’s are identical in this word. By similar reasoning to the previous example, we see that we have overcounted by the product of the number of rearrangements of three things times the number of possible rearrangements of two things $3! \times 2!$. Therefore, the number of five-letter words obtainable by rearranging the letters in “DADDA” is

$$\frac{5!}{3!\,2!} = \frac{120}{6 \times 2} = 10.$$

General Formula for Permutations with Repetitions

The number of permutations of n objects of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

1 Nickel and 3 Pennies The number of ways to get this outcome is the number of permutations of the word "NPPP" which is $\frac{4!}{3!} = 4$.

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

1 Nickel and 3 Pennies The number of ways to get this outcome is the number of permutations of the word "NPPP" which is $\frac{4!}{3!} = 4$.

2 Nickels and 2 Pennies The number of ways to get this outcome is the number of permutations of the word "NNPP" which is $\frac{4!}{2!\,2!} = 6$.

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

1 Nickel and 3 Pennies The number of ways to get this outcome is the number of permutations of the word “NPPP” which is $\frac{4!}{3!} = 4$.

2 Nickels and 2 Pennies The number of ways to get this outcome is the number of permutations of the word “NNPP” which is $\frac{4!}{2!\,2!} = 6$.

3 Nickels and 1 Penny The number of ways to get this outcome is the number of permutations of the word “NNNP” which is $\frac{4!}{3!} = 4$.

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

1 Nickel and 3 Pennies The number of ways to get this outcome is the number of permutations of the word "NPPP" which is $\frac{4!}{3!} = 4$.

2 Nickels and 2 Pennies The number of ways to get this outcome is the number of permutations of the word "NNPP" which is $\frac{4!}{2!\,2!} = 6$.

3 Nickels and 1 Penny The number of ways to get this outcome is the number of permutations of the word "NNNP" which is $\frac{4!}{3!} = 4$.

1 Nickel, 1 Dime, and 2 Pennies The number of ways to get this outcome is the number of permutations of the word "NDPP" which is $\frac{4!}{2!} = 12$.

Back to Problem 6: Counting Successful Outcomes

4 pennies There is only 1 way to get this outcome: (P,P,P,P).

1 Nickel and 3 Pennies The number of ways to get this outcome is the number of permutations of the word "NPPP" which is $\frac{4!}{3!} = 4$.

2 Nickels and 2 Pennies The number of ways to get this outcome is the number of permutations of the word "NNPP" which is $\frac{4!}{2!\,2!} = 6$.

3 Nickels and 1 Penny The number of ways to get this outcome is the number of permutations of the word "NNNP" which is $\frac{4!}{3!} = 4$.

1 Nickel, 1 Dime, and 2 Pennies The number of ways to get this outcome is the number of permutations of the word "NDPP" which is $\frac{4!}{2!} = 12$.

1 Dime and 3 Pennies The number of ways to get this outcome is the number of permutations of the word "DPPP" which is $\frac{4!}{3!} = 4$.

Problem 6 Conclusion

$$\text{Prob(Successful Outcome)} = \frac{\text{\# of Sucessful Outcomes}}{\text{Total \# of Outcomes}}$$

$$= \boxed{\frac{31}{256}}$$

Problem 7

The arithmetic mean of 10 consecutive even integers is 3. What is the least of these 10 even integers?

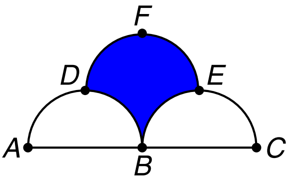
Problem 7

The arithmetic mean of 10 consecutive even integers is 3. What is the least of these 10 even integers?

If the mean is 3, then five of the numbers must be less than 3 and five greater than 3. The numbers less than 3 must be -6, -4, -2, 0, 2 and those greater than 3 must be 4, 6, 8, 10, and 12. The least of these is $\boxed{-6}$.

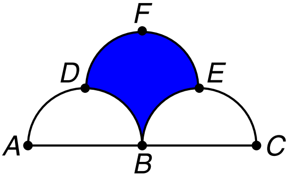
Problem 8

In the figure shown, arc ADB and arc BEC are semicircles, each with a radius of one unit. Points D, E, and F are the midpoints of arc ADB, arc BEC and arc DFE, respectively. If arc DFE is also a semicircle, what is the area of the shaded region?

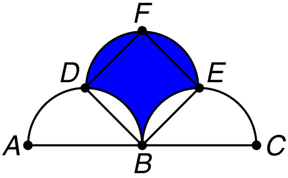


Problem 8

In the figure shown, arc ADB and arc BEC are semicircles, each with a radius of one unit. Points D, E, and F are the midpoints of arc ADB, arc BEC and arc DFE, respectively. If arc DFE is also a semicircle, what is the area of the shaded region?



Note that square $BEFD$ has the same area as the original shaded region, since the two half-football regions not shaded in the square are congruent to the two shaded half-football regions outside the square. Since the diagonal BF of the square is $d = 2$ units long, the square has area



$$\text{Shaded Area} = \frac{1}{2}d^2 = \frac{1}{2} \cdot 4 = \boxed{2}$$

Problem 9

Sue owns 11 pairs of shoes: six identical black pairs, three identical brown pairs and two identical gray pairs. If she picks two shoes at random, what is the probability that they are the same color and that one is a left shoe and the other is a right shoe? Express your answer as a common fraction.

Problem 9

Sue owns 11 pairs of shoes: six identical black pairs, three identical brown pairs and two identical gray pairs. If she picks two shoes at random, what is the probability that they are the same color and that one is a left shoe and the other is a right shoe? Express your answer as a common fraction.

There are 22 shoes.

Success with Black

Let's first consider the probability of drawing a pair of black shoes. There are 12 black shoes so the probability of drawing a black shoe on the first pick is $12/22 = 6/11$. For the second pick, there are 6 appropriate black shoes of 21 shoes left, so the chance of success on the second pick is $6/21$. Therefore, the probability of success with black shoes is

$$\frac{6}{11} \times \frac{6}{21} = \frac{36}{231}$$

Problem 9, Continued

Success with Brown

There are 6 brown shoes so the probability of drawing a brown on the first pick is $6/22 = 3/11$. For the second pick, there are 3 matching brown shoes and 21 shoes left so the chance of success on this pick is $3/21$ and the overall probability of success with brown is

$$\frac{3}{11} \times \frac{3}{21} = \frac{9}{231}$$

Problem 9, Continued

Success with Brown

There are 6 brown shoes so the probability of drawing a brown on the first pick is $6/22 = 3/11$. For the second pick, there are 3 matching brown shoes and 21 shoes left so the chance of success on this pick is $3/21$ and the overall probability of success with brown is

$$\frac{3}{11} \times \frac{3}{21} = \frac{9}{231}$$

Success with Gray

There are 4 gray shoes, so the chance of success on the first pick is $4/22 = 2/11$. For the second pick, there are 21 shoes remaining and 2 matching gray shoes, so the probability of sucess on this pick is $2/21$ and the overall probability of success with gray is

$$\frac{2}{11} \times \frac{2}{21} = \frac{4}{231}$$

Problem 9, Continued

The probability of success is the sum of the probability of success for black, gray, and brown:

$$\frac{36}{231} + \frac{9}{231} + \frac{4}{231} = \frac{49}{231} = \boxed{\frac{7}{33}}$$

Problem 10

Each student works at the same speed. If five students can complete a job in six days, how many days would it take three students to complete the same job?

Problem 10

Each student works at the same speed. If five students can complete a job in six days, how many days would it take three students to complete the same job?

We know that 3 students will take longer to do the job than 5 students. In fact the time needed is inversely proportional to the number of students assigned:

$$\text{Days Required} = 6 \times \frac{5}{3} = \boxed{10}$$