Warm-Up 13 Solutions

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Problem 1

Out of 28 students in Mr. Sullivan's homeroom, the ratio of boys to girls is 3:4. The ratio of students who have returned their field trip forms to those who have not is 3:1. If exactly half of the boys returned their field trip forms, how many girls have not returned their forms?

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Let B be the number of boys. Then $28 - B$ is the number of girls and the ratio of boys to girls is $\frac{B}{28-B} = \frac{3}{4}$ so that

$$4B = 3(28 - B) = 84 - 3B \implies 7B = 84 \implies B = \frac{84}{7} = 12$$

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and the number of girls is $28 - 12 = 16$. 6 (half) of the boys returned their forms and 6 (half) did not. Let G be the number of girls who did not return forms; then $(16 - G)$ did return forms. We are given the ratio of students who returned forms to those who didn't:

$$3 = \frac{6 + 16 - G}{6 + G} = 3 \implies 3(6 + G) = 18 + 3G = 22 - G \implies 4G = 4$$

so that $G = \boxed{1}$.

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$$3 \times 2 \times 1 = 6 = {}_3P_3.$$

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Let the numbers be a and b. We are told that $a + b = 15$ and $ab = 16$. We are asked to find

$$\frac{1}{a} + \frac{1}{b}$$

One way is to solve for a and b separately. But there is an easier way:

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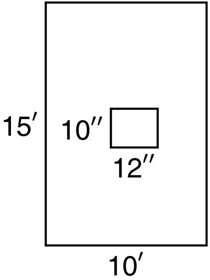
Problem 4

A 10-inch by 12-inch picture is to be enlarged and painted on a 15-foot by 10-foot wall. If the picture remains proportional (not distorted) and maintains the same orientation (not rotated), what is the greatest fraction of the wall that could be covered by the painting? Express your answer as a common fraction.



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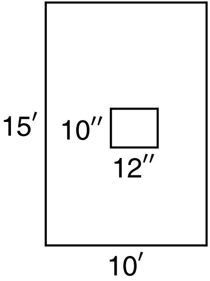

If the vertical dimension of the picture was expanded to 15′, then the horizontal dimension would be $12/10 \times 15' = 18'$ which is larger than the wall. So, instead, we expand the horizontal dimension of the picture to 10′ so that the vertical dimension is $10/12 \times 10' = 8\frac{1}{3}'$. The fraction of the wall that is covered up is then

$$\frac{8\frac{1}{3}'}{15'} = \frac{25}{45} = \boxed{\frac{5}{9}}.$$

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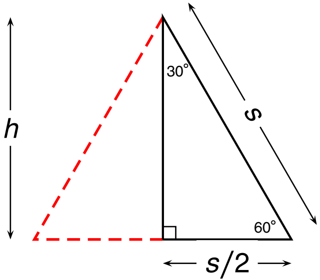
Let's put both equations in slope-intercept form:

$$2y - 2a = 6x \implies y = 3x + a, \quad y + 1 = (a + 6)x \implies y = (a + 6)x - 1$$

The slopes of the two parallel lines must be equal:

$$3 = a + 6 \implies a = \boxed{-3}$$

Aside: Altitude and Area of an Equilateral Triangle



Recall that the interior angles of an equilateral triangle measure 60°. Given the sidelength s, then by Pythagoras, the altitude h is found as

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2 \implies h^2 = \frac{3}{4}s^2 \implies \boxed{h = \frac{\sqrt{3}}{2}s}$$

and the area of the equilateral triangle is

$$A = \frac{1}{2}sh = \frac{1}{2}s\frac{\sqrt{3}}{2}s = \boxed{\frac{\sqrt{3}}{4}s^2}$$

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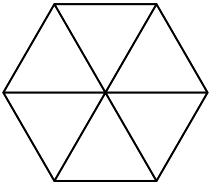
Partition the regular hexagon into 6 congruent, equilateral triangles of side length $s = 6$. Then the area of the hexagon is six times the area of a single triangle:



$$
\begin{aligned}
A = 6A_{\Delta} &= 6 \times \frac{1}{2}sh \\
&= 3sh = 3s\frac{\sqrt{3}}{2}s \\
&= \frac{3\sqrt{3}}{2}s^2 = \frac{3\sqrt{3}}{2}6^2 \\
&= \boxed{54\sqrt{3}}
\end{aligned}
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One could imagine trying to count the number of combinations of 7 items taken 3 at a time, but many of these combinations will result in the same sum. Instead, consider the least and greatest sum that can be formed. These are $-2 - 1 + 1 = -2$ and $3 + 4 + 5 = 12$, respectively. Since all integers between these can also be formed as sums, then the number of distinct sums is

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$$12 - (-2) + 1 = \boxed{15}$$

Problem 8

A two-pan balance scale comes with a collection of weights. Each weight weighs a whole number of grams. Weights can be put in either or both pans during a weighing. To ensure any whole number of grams up to 100 grams can be measured, what is the minimum number of weights needed in the collection?

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If we were only allowed to put weights in one of the pans, then the following set of weights would be the fewest needed: $\{1, 2, 4, 8, 16, 32, 64\}$, since we can represent any whole number less than 128 by counting in base 2 using 7 binary digits. For example, 100 = 64 + 32 + 4.

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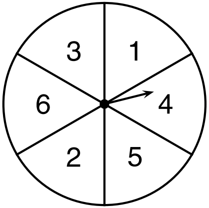
If we can put weights in either pan, then we can use the sum or difference of our weights to measure whole numbers of grams. This means we can count in base 3! The only weights needed are 1, 3, 9, 27, and 81 grams, for a total of $\boxed{5}$ weights.

Problem 9

Ann has the spinner shown to the right. The six sectors of the spinner are congruent. What is the probability of spinning integers with a positive difference of 1 on her first two spins? Express your answer as a common fraction.



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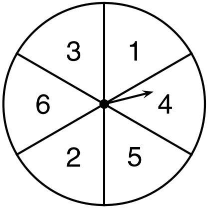

The number of outcomes for the first spin is 6, and similarly for the second spin. So the number of ordered pairs (m, n) that could occur for the combined spins is $6 \times 6 = 36$.

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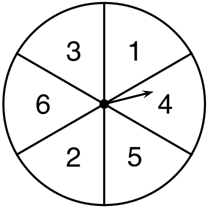

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The possible successful outcomes are (1,2), (2,1), (2,3), (3,2), (3,4), (4,3), (4,5), (5,4), (5,6), and (6,5), of which there are 10. So the probability of a sucessful outcome is

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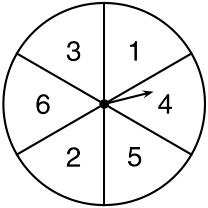

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$$\frac{10}{36} = \boxed{\frac{5}{18}}$$

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Since $7a + 12b = 1$, then $b = (1 - 7a)/12$, and

$$a + b = a + \frac{1 - 7a}{12} = \frac{12a + 1 - 7a}{12} = \frac{5a + 1}{12} < 2005$$

Now try some guess and check: Try $a + b = 2004$. Then

$$\frac{5a + 1}{12} = 2004 \implies 5a = (12)2004 - 1 = 24047$$

which will not yield an integer for a when we divide both sides by 5. Try $a + b = \boxed{2003}$. Then

$$\frac{5a + 1}{12} = 2003 \implies 5a = (12)2003 - 1 = 24035$$

which gives $a = 4807$, $b = -2804$.