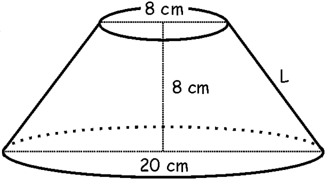
Warm-Up 12 Solutions

Peter S. Simon

December 8, 2004

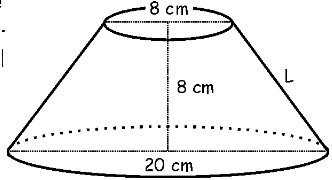
Problem 1

The lateral surface area of the frustum of a solid right cone is the product of the slant height L and the average circumference of the two circular faces. What is the number of square centimeters in the total surface area of the frustum shown here? Express your answer in terms of π.

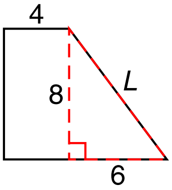


Problem 1

The lateral surface area of the frustum of a solid right cone is the product of the slant height L and the average circumference of the two circular faces. What is the number of square centimeters in the total surface area of the frustum shown here? Express your answer in terms of π.



We can find L from the Pythagorean theorem: $L = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$ so that the total surface area is



$$\begin{aligned} A &= \pi r_1^2 + \pi r_2^2 + \frac{2\pi r_1 + 2\pi r_2}{2} \times L \\ &= \pi(4^2 + 10^2) + (4\pi + 10\pi) \times 10 \\ &= 116\pi + 140\pi = \boxed{256\pi} \end{aligned}$$

Problem 2

In a two-day period, a committee drank 14 cans of soda. How many cans of soda would the committee drink in a week if they continue drinking at the same rate?

Problem 2

In a two-day period, a committee drank 14 cans of soda. How many cans of soda would the committee drink in a week if they continue drinking at the same rate?

The amount of soda consumed is proportional to the length of time:

$$\#\text{ Cans} = \frac{7}{2} \times 14 = \boxed{49}$$

Problem 3

The function $f(n) = n^2 + n + 17$ for $0 \leq n \leq 15$ generates prime numbers. What is the value of $f(10) - f(9)$?

Problem 3

The function $f(n) = n^2 + n + 17$ for $0 \le n \le 15$ generates prime numbers. What is the value of $f(10) - f(9)$?

$$f(10) = 10^2 + 10 + 17 = 100 + 27 = 127$$
$$f(9) = 9^2 + 9 + 17 = 81 + 26 = 107$$
$$f(10) - f(9) = 127 - 107 = \boxed{20}$$

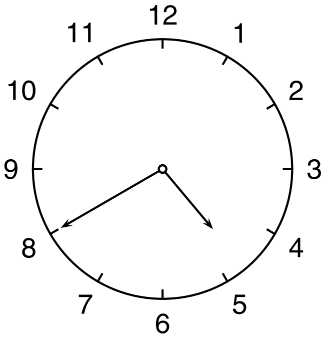
Problem 4

The time is 4:40 p.m. What is the degree measure of the smallest angle formed by the minute hand and the hour hand?

Problem 4

The time is 4:40 p.m. What is the degree measure of the smallest angle formed by the minute hand and the hour hand?

Each pair of adjacent numbers on the clock is separated by 30°.

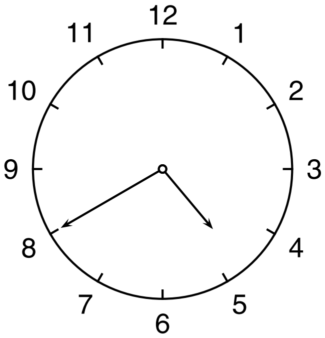


Problem 4

The time is 4:40 p.m. What is the degree measure of the smallest angle formed by the minute hand and the hour hand?

Each pair of adjacent numbers on the clock is separated by 30°.

The minute hand points to the 8 on the clock.



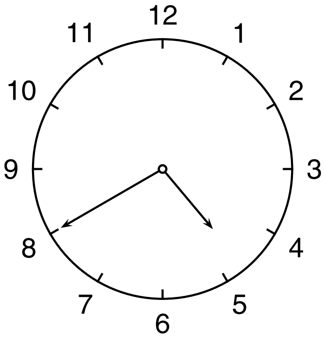
Problem 4

The time is 4:40 p.m. What is the degree measure of the smallest angle formed by the minute hand and the hour hand?

Each pair of adjacent numbers on the clock is separated by 30°.

The minute hand points to the 8 on the clock.

The hour hand is $40/60 = 2/3$ of the way from the 4 to the 5 position.



Problem 4

The time is 4:40 p.m. What is the degree measure of the smallest angle formed by the minute hand and the hour hand?

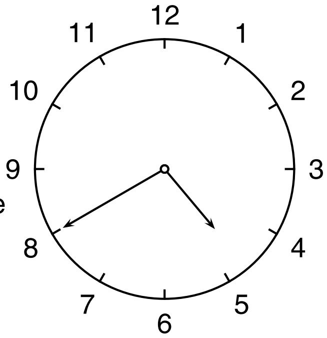
Each pair of adjacent numbers on the clock is separated by 30°.

The minute hand points to the 8 on the clock.

The hour hand is 40/60 = 2/3 of the way from the 4 to the 5 position.

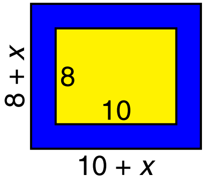
The angle between the hour and minute hands is then

$$3 \times 30 + \frac{1}{3} \times 30 = 90 + 10 = \boxed{100^\circ}$$



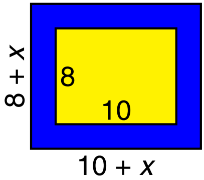
Problem 5

An 8×10 picture is placed on top of a larger rectangular mat so that there is a border of the same width along each side of the picture. If the area of the border around the picture is 88, what is the outside perimeter of the entire mat?



Problem 5

An 8×10 picture is placed on top of a larger rectangular mat so that there is a border of the same width along each side of the picture. If the area of the border around the picture is 88, what is the outside perimeter of the entire mat?



The border (blue) area is the difference between the areas of the outer and inner rectangles:

$$
\begin{aligned}
88 &= (x+10)(x+8) - 10 \times 8 = x^2 + 18x + 80 - 80 \\
&= x^2 + 18x
\end{aligned}
$$

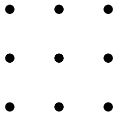
$$0 = x^2 + 18x - 88 = (x-4)(x+22)$$

So the possible solutions are $x = 4$ and $x = -22$. Since the negative solution doesn't make any sense, we choose $x = 4$, so that the perimeter of the mat is

$$2(10+4) + 2(8+4) = 2(14+12) = 2(26) = \boxed{52}$$

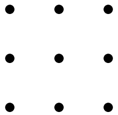
Problem 6

How many triangles are there whose three vertices are points on this square 3×3 grid?



Problem 6

How many triangles are there whose three vertices are points on this square 3×3 grid?



To avoid drawing all the possible triangles, let's start by counting the number of ways we can choose a set containing 3 of the 9 vertices:

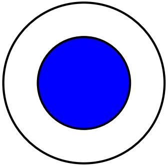
$$ {}_9C_3 \equiv \binom{9}{3} = \frac{9!}{3!\,(9-3)!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 3 \times 4 \times 7 = 84 $$

However, not all sets of three vertices define a triangle. There are 3 horizontal lines, three vertical lines, and 2 diagonal lines, so the number of triangles is

$$ 84 - 3 - 3 - 2 = 84 - 8 = \boxed{76} $$

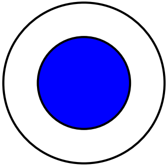
Problem 7

The students of LPSS are planning a carnival. One of the contests will be a modified dart throw. The radius of the entire round dartboard (made of two concentric circles) is eight inches. What is the radius of the shaded circle if the area of the non-shaded region is three times the area of the shaded region?



Problem 7

The students of LPSS are planning a carnival. One of the contests will be a modified dart throw. The radius of the entire round dartboard (made of two concentric circles) is eight inches. What is the radius of the shaded circle if the area of the non-shaded region is three times the area of the shaded region?



Let $R = 8$ and be the radius of the large cicle, and r be the radius of the small, shaded circle. The area of the shaded region is πr^2. The area of the non-shaded region is $\pi(R^2 - r^2) = \pi(64 - r^2)$. The ratio of these areas is

$$3 = \frac{\pi(64 - r^2)}{\pi r^2} = \frac{64 - r^2}{r^2} = \frac{64}{r^2} - 1$$

so that

$$\frac{64}{r^2} = 4 \implies r^2 = \frac{64}{4} = 16 \implies r = \boxed{4}$$

Problem 8

A magician designed an unfair coin so that the probability of getting a Head on a flip is 60%. If he flips the coin three times, what is the probability that he flips more Heads than Tails? Express your answer as a common fraction.

Problem 8

A magician designed an unfair coin so that the probability of getting a Head on a flip is 60%. If he flips the coin three times, what is the probability that he flips more Heads than Tails? Express your answer as a common fraction.

The probability of getting a Head on any flip is $\frac{3}{5}$. The probability of getting a Tail on any flip is $\frac{2}{5}$. The outcomes that result in more heads than tails and their associated probabilities are

Outcome	**Probability**
HHT	$\frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} = \frac{18}{125}$
HTH	$\frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{18}{125}$
THH	$\frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} = \frac{18}{125}$
HHH	$\frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} = \frac{27}{125}$

So the probability that more heads than tails occurred is:

$$P = 3 \times \frac{18}{125} + \frac{27}{125} = \boxed{\frac{81}{125}}$$

Problem 9

There exists a two-digit integer, “AB,” such that $AB = 2^{(A+B)}$. What is the value of the product $A \times B$?

Problem 9

There exists a two-digit integer, “AB,” such that $AB = 2^{(A+B)}$. What is the value of the product $A \times B$?

The possible two-digit powers of 2 are 16, 32, and 64. Trial and error shows that

$$2^{3+2} = 2^5 = 32$$

so $A = 3$, $B = 2$, and

$$A \times B = 3 \times 2 = \boxed{6}$$

Problem 10

A car radiator has a 16-quart capacity and is currently filled with a 40% antifreeze solution. How many quarts of this solution should be drained off and replaced with 100% antifreeze to obtain a 50% antifreeze solution? Express your answer as a common fraction.

Problem 10

A car radiator has a 16-quart capacity and is currently filled with a 40% antifreeze solution. How many quarts of this solution should be drained off and replaced with 100% antifreeze to obtain a 50% antifreeze solution? Express your answer as a common fraction.

The number of quarts of pure antifreeze currently in the radiatior is $0.4 \times 16 = 6.4$. The total amount needed in the radiator to make a 50% solution is 8 quarts. Let X be the number of quarts removed and replaced with 100% solution. Then the amount of pure antifreeze left in the tank after removing X quarts is $0.4 \times (16 - X)$. The amount in the tank after adding back in X quarts of 100% solution is

$$8 = 0.4 \times (16 - X) + X = 6.4 - 0.4X + X = 6.4 + 0.6X$$

$$0.6X = 1.6 \implies X = \frac{1.6}{0.6} = \frac{0.8}{0.3} = \frac{8}{3}$$