Warm-Up 11 Solutions

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Problem 1

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Since there are 10 bottles and one bottle of water, the probability is $\boxed{\frac{1}{10}}$.

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An isosceles triangle has two sides with the same length. The candidates for side lengths are 2, 3, 5, 7, 11, 13, 17, and 19. Note that $5 + 5 + 13 = 23$ but these can not be the side lengths of a triangle (why?). Experimentation shows that the smallest perimeter is obtained with lengths 11, 11, and 2, which sum to $\boxed{24}$.

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Method I

To make the larger of the two numbers as large as possible, let's try writing it as "99x" or $990 + x$ where x is the unknown 1's digit. The smaller of the two numbers will be "x99" or $100x + 99$ and the difference will be

$$990 + x - (100x + 99) = 990 - 99 + (x - 100x) = 891 - 99x = 396$$

$$99x = 891 - 396 = 495 \implies x = \frac{495}{99} = 5$$

So the original number is $\boxed{995}$ and its mirror image is 599. Their difference is indeed $995 - 599 = 396$, as required.

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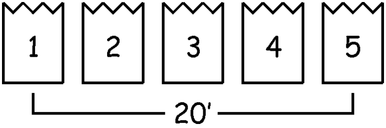
Method II

To make the larger of the two numbers as large as possible, let's try writing it as "99x" where x is the unknown 1's digit. The smaller of the two numbers will be "x99". Since the 1's digit of the difference must be 6, x must be 5. So the original number is $\boxed{995}$ and its mirror image is 599. Their difference is indeed $995 - 599 = 396$, as required.

Problem 4

Luminary bags are equally spaced in a row along a straight, long road. There are 20 feet from the first bag to the fifth bag, when measured as shown. How many feet are there from the first bag to the 25th bag?



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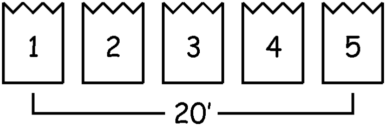

As shown, there are four center-to-center increments that measure 20 feet, so a single center-to-center distance is 5 feet. For 25 bags, there will be 24 increments, so the distance is $24 \times 5 = \boxed{120}$ feet.

Problem 5

In the following addition problem, each distinct letter represents a different digit from 1 through 9. What is the greatest possible value of the four-digit number represented by the word “EXAM”?

$$
\begin{array}{rcccc}
 & T & A & K & E \\
+ & H & O & M & E \\
\hline
 & E & X & A & M
\end{array}
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To maximize the sum we set E = 9, which means that M = 8 (from the 1’s column).

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To maximize the sum we set E = 9, which means that M = 8 (from the 1's column). We now have TAK9 + HO89 = 9XA8. Since the 8 has already been used, let's try 7 for X:

$$\begin{array}{r} T\quad A\quad K\quad 9 \\ +\,H\quad O\quad 8\quad 9 \\ \hline 9\quad 7\quad A\quad 8 \end{array}$$

We know that $A \neq 6$ because otherwise, the tens column of the sum shows that K would have to be 7, which has already been used.

Problem 5, Continued

$$\begin{array}{rcccc} & T & A & K & 9 \\ + & H & O & 8 & 9 \\ \hline & 9 & 7 & A & 8 \end{array}$$

Case 1: A = 5

Then K is 6 and we're carrying a 1 to the hundreds column. Since $1 + 5 + O = 7$ in the hundreds column, then O is 1. We can not make 9 from the sum of T and H in the thousands column, because the digits 9, 8, 7, 6, and 5 have already been used. We conclude that $A \neq 5$.

Case 2: A = 4

Then K is 5 and we're carrying a 1 to the hundreds column. Since $1 + 4 + O = 7$ in the hundreds column, then O is 2. We can make 9 from the sum of $T = 6$ and $H = 3$.

The value of EXAM is maximized at $\boxed{9748}$.

Problem 6

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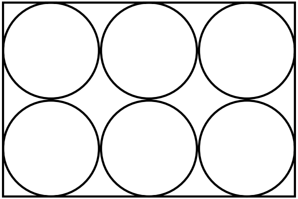
No prime number is abundant since the only proper factor of a prime number is 1. (A proper factor of a number is a factor which is less than the m number itself.) We start by examining 4, the smallest composite number not a multiple of 6.

Number	Sum of Proper Factors
4	$1 + 2 = 3 < 4$
8	$1 + 2 + 4 = 7 < 8$
10	$1 + 2 + 5 = 8 < 10$
14	$1 + 2 + 4 + 7 = 14$
15	$1 + 3 + 5 = 9 < 15$
16	$1 + 2 + 4 + 8 = 15 < 16$
20	$1 + 2 + 4 + 5 + 10 = 22 > 20$

So $\boxed{20}$ is the smallest abundant number that is not a multiple of 6.

Problem 7

The sum of the area of the six congruent circles is 150π square centimeters. What is the area of the rectangle?



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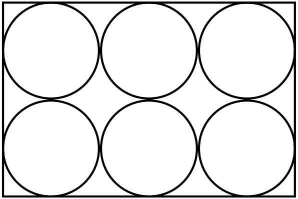

The total area occupied by the circles is

$$6\pi r^2 = 150\pi \implies r^2 = \frac{150}{6} = 25 \implies r = 5$$

where r is the radius of the circles. The area of the rectangle is

$$A = 6r \times 4r = 30 \times 20 = \boxed{600}$$

Problem 8

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Let the three integers be n, $n+1$, and $n+2$. Then

$$n(n+1)(n+2) = 33(n+n+1+n+2)$$
$$n(n+1)(n+2) = 33(3n+3) = 99n+99 = 99(n+1)$$
$$n(n+2) = 99$$

We can guess and check that $n = 9$, $n+2 = 11$ works, so the sum of the three integers is $9 + 10 + 11 = \boxed{30}$.

Problem 9

Jack is tiling his patio with concrete pavers that have a pattern in which six congruent rectangles are arranged. One such paver is shown here. If the area of the paver is 600 square inches, what is its perimeter?

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If L and W are the length and width of a single rectangle, then $L = 4W$ and the total area of the six rectangles is

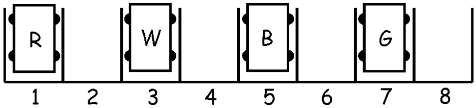
$$6LW = 600 \implies LW = 100 \implies 4W^2 = 100 \implies W^2 = 25$$

so that $W = 5$ and $L = 4W = 20$.
The perimeter of the paver is then

$$P = 4L + 4W = 4(L + W) = 4(20 + 5) = \boxed{100}$$

Problem 10

A parking lot has a row of eight parking spaces numbered sequentially 1 through 8. Four cars (red, white, blue and green) are parked such that no two cars are in adjacent parking spots. One such arrangement is shown here. How many arrangements are possible?



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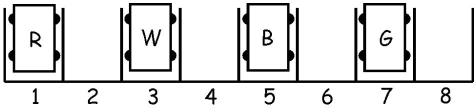

First, we make a list of the possible parking positions that could be occupied: 1357, 1358, 1368, 1468, 2468.
So there are 5 possible selections of parking spaces. For each of these selections, the four cars can be arranged in $4! = 4 \cdot 3 \cdot 2 = 24$ different ways. So the total number of ways the cars can be parked is $5 \times 24 = \boxed{120}$.