# Warm-Up 10 Solutions

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# Problem 1

Nine ping-pong balls are numbered 1 through 9. How many different combinations of three balls have a sum of 16?

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To be methodical it is best to make a table of possible triples. We will list the triples in increasing order:

$$
\begin{aligned}
&1+6+9\\
&1+7+8\\
&2+5+9\\
&2+6+8\\
&3+4+9\\
&3+5+8\\
&3+6+7\\
&4+5+7
\end{aligned}
$$

So there are $\boxed{8}$ combinations.

# Problem 2

If 2 Lops are equal to 4 Gops, then 3 Lops are equal to how many Gops?

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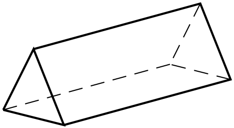
$$3\,\text{Lops} \times \frac{4\,\text{Gops}}{2\,\text{Lops}} = \boxed{6\,\text{Lops}}$$

# Problem 3

One of the five faces of the triangular prism shown here will be used as the base of a new pyramid. The numbers of exterior faces, vertices and edges of the resulting shape (the fusion of the prism and pyramid) are added. What is the maximum value of this sum?



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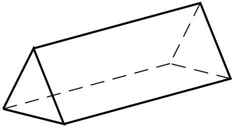
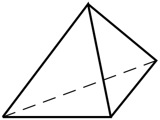
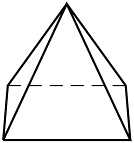


A pyramid is a solid figure which has a polygon for its single base and triangles for its sides (or faces). Examples:



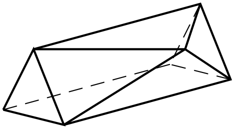
Triangular Pyramid



Rectangular Pyramid

# Problem 3, Continued

It is clear that adding a rectangular pyramid to one of the rectangular faces of the prism will result in a greater total of faces, edges, and vertices:



| | |
|---|---|
| | Faces |
| | Edges |
| | Vertices |
| | Total |

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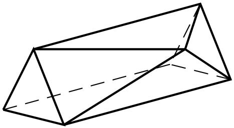

| | |
|---|---|
| 8 | Faces |
| | Edges |
| | Vertices |
| | Total |

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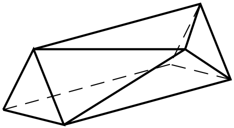

| | |
|---|---|
| 8 | Faces |
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| | Vertices |
| | Total |

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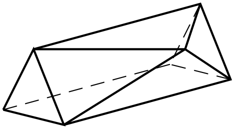

| | |
|---|---|
| 8 | Faces |
| 13 | Edges |
| 7 | Vertices |
| | Total |

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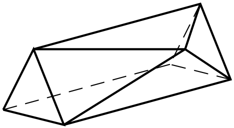

| | |
|---|---|
| 8 | Faces |
| 13 | Edges |
| 7 | Vertices |
| **28** | Total |

# Problem 4

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$$\boxed{19}$$

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We can include 0 as a digit since we don't have to try dividing by 0. So the smallest five-digit number to try is $10,234$ which is divisible by 1 and 2, but not by 3 (since $1 + 2 + 3 + 4 = 10$) and not by 4 (since 34 is not divisible by 4).

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If we increase this number by 1, the last digit is 5 and it won't be divisible by 2.

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Adding one more yields $\boxed{10{,}236}$ which is divisible by all of its digits.

# Problem 6

Marion’s first nine quiz scores are 77, 85, 79, 92, 86, 92, 76, 97 and 81. If the scores of her next two quizzes change the mode of the 11 scores to a different single value and decrease the mean, what is the difference between the greatest possible mean of the last two quizzes and the least possible mean of the last two quizzes?

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Sorting the scores into increasing order yields 76, 77, 79, 81, 85, 86, 92, 92, 97. The mode is the most frequently occuring score:

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Sorting the scores into increasing order yields 76, 77, 79, 81, 85, 86, 92, 92, 97. The mode is the most frequently occuring score: 92. The mean is average score:

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Sorting the scores into increasing order yields 76, 77, 79, 81, 85, 86, 92, 92, 97. The mode is the most frequently occuring score: 92. The mean is average score: 85.

To change the mode to a single number means that the new mode occurs three times. Therefore both new scores are identical to one of the previous scores less than 85. Their minimum mean is 76 and their maximum mean is 81. The difference is $\boxed{5}$.

# Problem 7

Define $A \& B$ as $A \& B = \frac{A+B}{2}$. What is the value of $(3 \& 5) \& 8$?

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$$(3 \,\&\, 5) \,\&\, 8 = \frac{3+5}{2} \,\&\, 8 = 4 \,\&\, 8 = \frac{4+8}{2} = \boxed{6}$$

# Problem 8

Darina has five sticks measuring 5 cm, 5 cm, 8 cm, 14 cm and 14 cm. Using exactly three sticks as the sides of a triangle, how many non-congruent triangles are possible if the sticks are joined only at their endpoints?

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For three sticks of length $a$, $b$, and $c$, the requirement that they can form a triangle is

$$c < a + b, \quad a < b + c, \quad b < c + a,$$

or: Any side must be shorter than the sum of the other two sides.

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The only triangles that can be constructed using the above criteria have side lengths (5,14,14), (8,14,14), and (8,5,5). So there are $\boxed{3}$ such triangles.

# Problem 9

Ann starts counting the letters of the alphabet beginning with A. When she gets to Z, she goes backwards from Y to A and then reverses again going from B to Z. If she continues this process, what is the 2005th letter that she will count?

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A is 1, B is 2, . . . , Z is 26, Y is 27, . . . , A is 51 (because Z was counted only once during the round trip). On the next trip Z is $A + 25 = 51 + 25 = 76$, and the next $A$ will be $Z + 25 = 76 + 25 = 101$. So A occurs at 1, 51, 101, . . . , 2001, so 2005 must be $\boxed{\text{E}}$.

# Problem 10

When simplified, what is the value of $\sqrt{3} \times 3^{\frac{1}{2}} + 12 \div 3 \times 2 - 4^{\frac{3}{2}}$ ?

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Note

$$3^{\frac{1}{2}} = \sqrt{3}$$

and

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8,$$

so

$$\begin{aligned}\sqrt{3} \times 3^{\frac{1}{2}} + 12 \div 3 \times 2 - 4^{\frac{3}{2}} &= \left(\sqrt{3} \times 3^{\frac{1}{2}}\right) + [(12 \div 3) \times 2] - 4^{\frac{3}{2}} \\ &= (\sqrt{3} \times \sqrt{3}) + (4 \times 2) - 8 \\ &= 3 + 8 - 8 = \boxed{3}\end{aligned}$$