Lecture 4: Geometry II

LPSS MATHCOUNTS

19 May 2004

Some Well-Known Pythagorean Triples

A *Pythagorean triple* is a set of three relatively prime[1] natural numbers a, b, and c satisfying $a^2 + b^2 = c^2$:

$$\boxed{3^2 + 4^2 = 5^2}$$

$$5^2 + 12^2 = 13^2$$

$$7^2 + 24^2 = 25^2$$

$$8^2 + 15^2 = 17^2$$

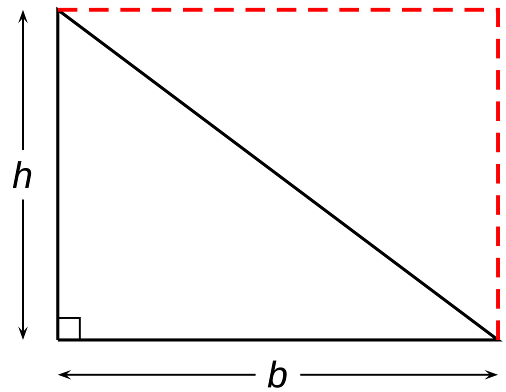
$$9^2 + 40^2 = 41^2$$

$$12^2 + 35^2 = 37^2 20^2 + 21^2 = 29^2$$

[1] No factors common to all three numbers other than 1

Area of a Right Triangle

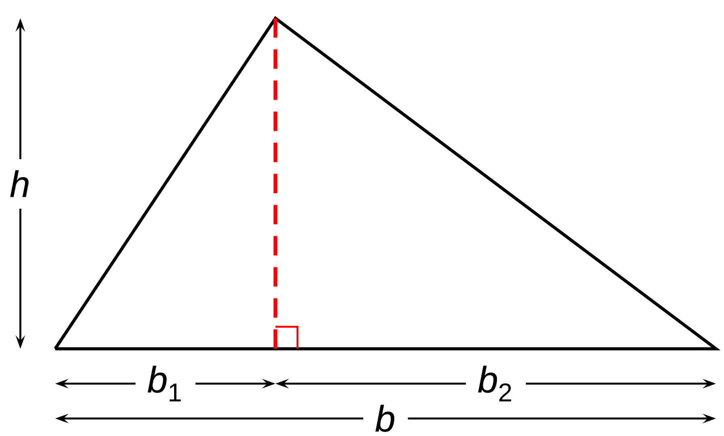
Recall that the area of a right triangle is easily found by considering the triangle to be half of a rectangle:



$$\text{Area} = \frac{1}{2}bh$$

Area of Arbitrary Triangles (Case 1)

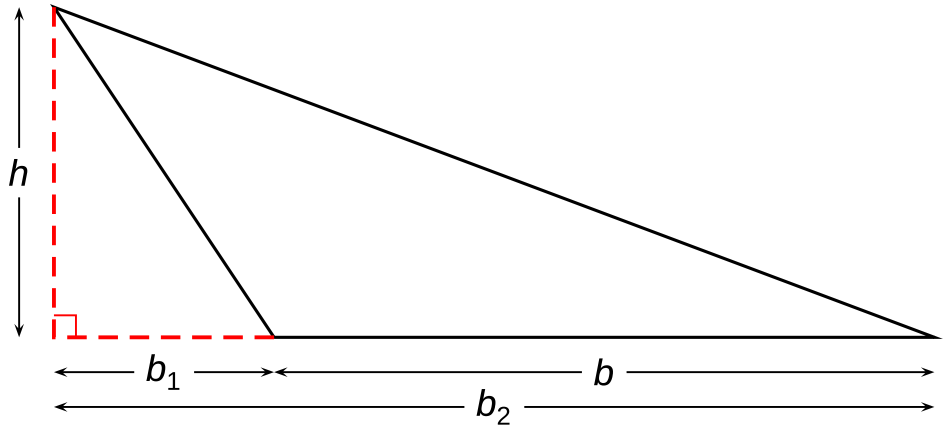
In this case, the vertex located above the “base” is in between the other two vertices:



$$\text{Area} = \text{Area}_1 + \text{Area}_2 = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}bh$$

Area of Arbitrary Triangles (Case 2)

In this case, the vertex located above the base is outside the projection of the base:



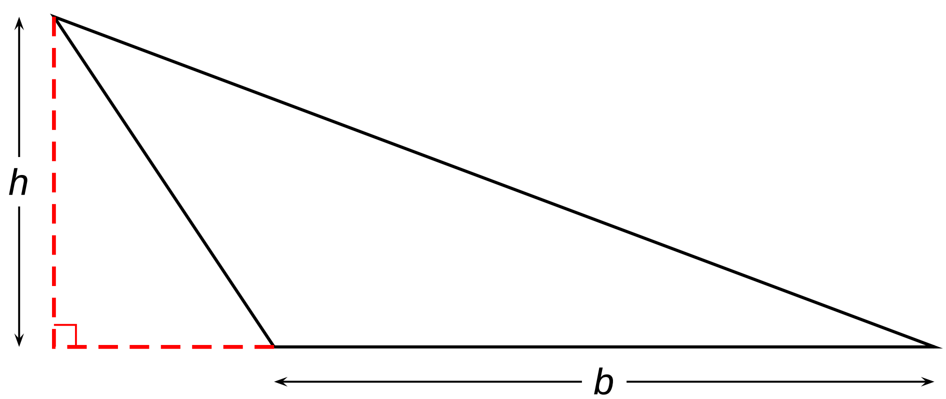
$$\text{Area} = \text{Area}_2 - \text{Area}_1 = \frac{1}{2}b_2h - \frac{1}{2}b_1h = \frac{1}{2}(b_2 - b_1)h = \frac{1}{2}bh$$

Area of Triangles—Summary

The area of **any** triangle is given by the formula

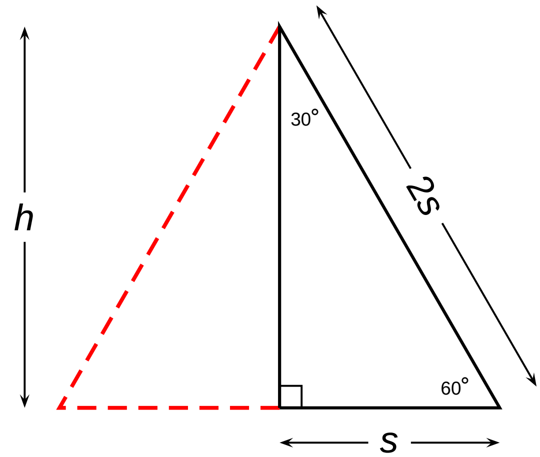
$$\text{Area} = \frac{1}{2}bh,$$

where b (the *base*) is the length of any side of the triangle, and h (the height or altitude) is the perpendicular distance measured from the line containing the base to the remaining vertex.



Special Triangles: The 30-60-90 Triangle

Recall that an equilateral triangle has interior angles of measure $180^\circ/3 = 60^\circ$. We can obtain a 30-60-90 right triangle by bisecting an equilateral triangle:



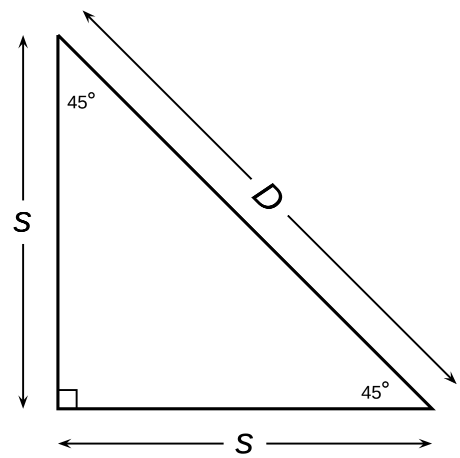
From Pythagoras: $h^2 + s^2 = (2s)^2 = 4s^2$

$$h = \sqrt{4s^2 - s^2} = \sqrt{3s^2} = \sqrt{3}s \approx 1.732s$$

$$\text{Area} = \frac{1}{2}sh = \frac{\sqrt{3}}{2}s^2$$

Special Triangles: The 45-45-90 or Right Isosceles Triangle

Recall that a square has interior angles of measure 90°. We can obtain a right isosceles triangle by bisecting a square along its diagonal:

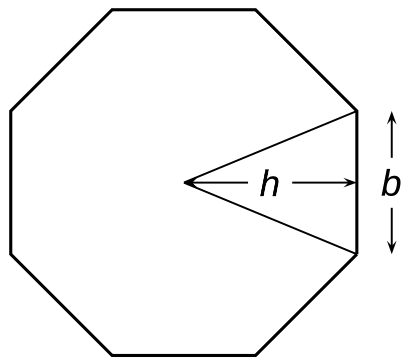


From Pythagoras: $D^2 = s^2 + s^2 = 2s^2$

$$D = \sqrt{2}s$$

The Area of a Circle

Recall that the circumference of a circle is $C = 2\pi r$, where r is the radius. Consider approximating a circle by an n-sided regular polygon:

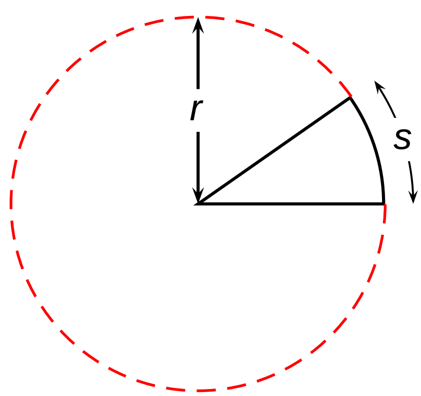


$$\text{Area of polygon} = \underbrace{\frac{1}{2}bh + \frac{1}{2}bh + \cdots + \frac{1}{2}bh}_{n \text{ terms}} = \frac{1}{2}h \underbrace{(b + b + \cdots + b)}_{\text{circumference of polygon}}$$

$$\xrightarrow[n\to\infty]{} \text{Area of Circle} = \frac{1}{2}rC = \frac{1}{2}r(2\pi r) = \boxed{\pi r^2}$$

The Area of a Circular Sector

We can find the area by noting that the area of the sector is proportional to the length of subtended arc:

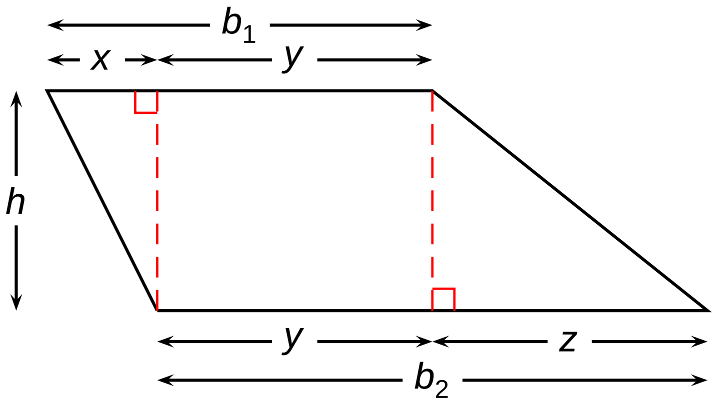


$$\text{Sector Area} = \frac{s}{2\pi r} \times (\text{Circle Area})$$
$$= \frac{s}{2\pi r}\pi r^2 = \frac{1}{2}sr$$

This is easy to remember because it is the same as a triangle of base s and height r.

The Area of a Trapezoid

A trapezoid is a quadrilateral with exactly one pair of opposite sides parallel. We find the area by decomposing the trapezoid into right triangles and a rectangle:

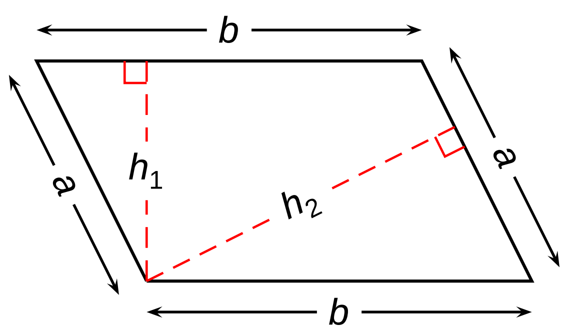


$$\text{Trapezoid area} = \frac{1}{2}xh + yh + \frac{1}{2}zh = \left(\frac{1}{2}xh + \frac{1}{2}yh\right) + \left(\frac{1}{2}yh + \frac{1}{2}zh\right)$$

$$= \frac{1}{2}(x+y)h + \frac{1}{2}(y+z)h = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \boxed{\frac{b_1+b_2}{2}h}$$

The Area of a Parallelogram

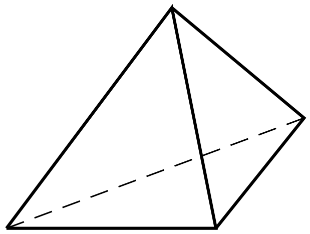
A parallelogram is a quadrilateral with exactly two pairs of opposite sides parallel (and congruent). We find the area using the same method as for the trapezoid. Note that any side of the parallelogram can be used as the "base."



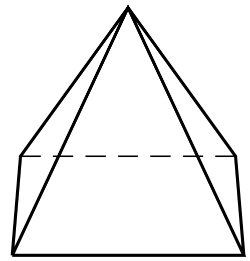
$$\boxed{\text{Parallelogram area} = bh_1 = ah_2}$$

Pyramids

A *pyramid* is a solid figure which has a polygon for its single *base* and triangles for its sides (or *faces*). Examples:



Triangular Pyramid



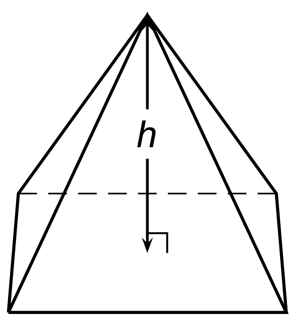
Rectangular Pyramid

Pyramid Surface Area and Volume

The **surface area** of a pyramid is found by summing the areas of all of the triangular faces, plus the area of the base.
The **volume** of a pyramid is given by the formula

$$V = \frac{1}{3}hA$$

where A is the area of the base, and h is the perpendicular height of the free vertex above the base. (Note: same formula works for a cone!)



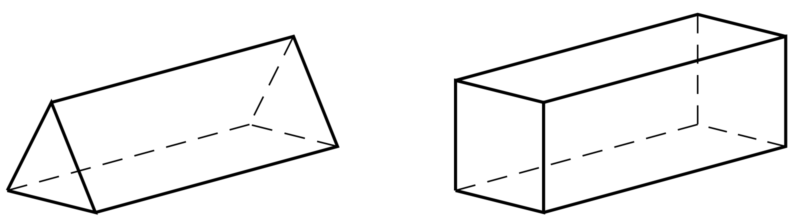
Pyramid Volume Example

A rectangular pyramid has a base measuring 6 inches by 10 inches. If the height is 40 inches, what is the volume of the pyramid in cubic inches?
Solution:

$$V = \frac{1}{3}hA = \frac{1}{3} \times 40 \times (6 \times 10) = 40 \times \frac{60}{3} = 40 \times 20 = 800 \text{ cubic inches.}$$

Prisms

A **prism** is a solid figure consisting of two identical, parallel bases, connected by faces consisting of rectangles:



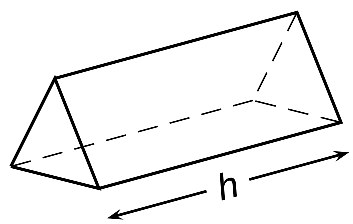
Triangular Prism Rectangular Prism

Prism Surface Area and Volume

The **surface area** of a prism is found by summing the areas of all of the rectangular faces, plus the area of the two identical bases.
The **volume** of a prism is given by the formula

$$V = hA$$

where A is the area of the base, and h is the perpendicular distance between the bases. (Note: same formula works for a circular cylinder!)



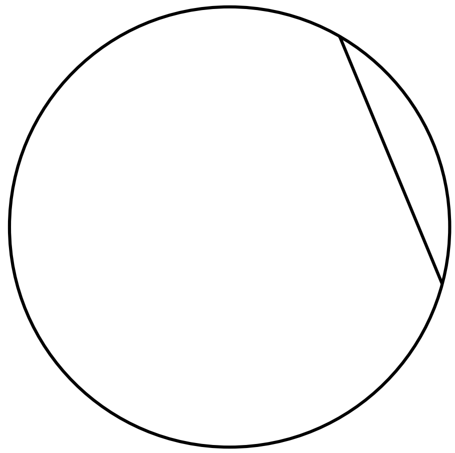
Prism Volume Example

A triangular prism has a right triangle base measuring 3 inches by 4 inches by 5 inches. If the height is 40 inches, what is the volume of the prism in cubic inches?
Solution:

$$V = hA = h \times \frac{1}{2} \times 3 \times 4 = 6h = 6 \times 40 = 240 \text{ cubic inches.}$$

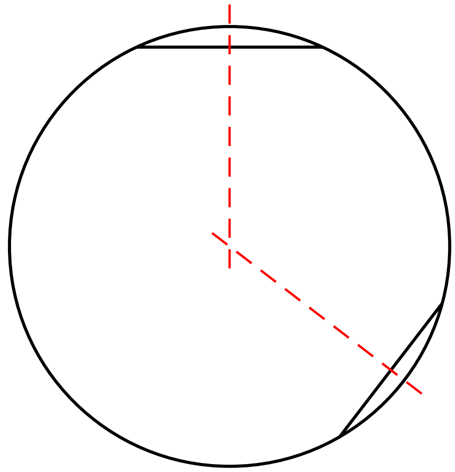
Chords of Circles

A *chord* is a line segment joining two points on a circle:



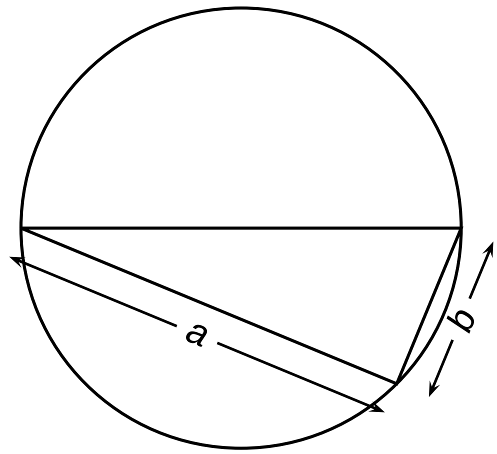
Chords of Circles (Cont.)

Note that the center of the circle lies on the *perpendicular bisector* of the chord. Thus, given two chords, we can find the center:



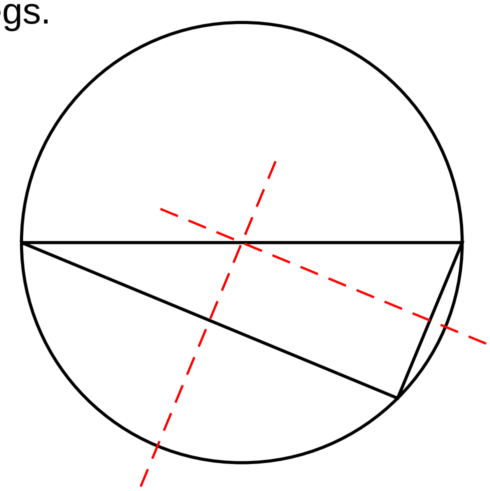
Right Triangle Inscribed in a Circle

We can find the center, and thus the radius, by locating the perpendicular bisectors of the two legs.



Right Triangle Inscribed in a Circle (Cont.)

We can find the center, and thus the radius, by locating the perpendicular bisectors of the two legs.



Note that we have constructed a rectangle of sides $a/2$ and $b/2$ whose diagonal is the circle's radius of length $r = D/2$, where D is the diameter of the circle.

Right Triangle Inscribed in a Circle (Cont.)

From Pythagoras, then

$$(a/2)^2 + (b/2)^2 = (D/2)^2 \implies a^2 + b^2 = D^2$$

so that the hypotenuse of the triangle must be a diameter of the circle!
Summary: Any right triangle inscribed in a circle forms a diameter of the circle with its hypotenuse!

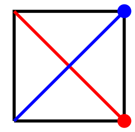
Some formulas for Area and Volume

1. Area of a triangle given the base b and height h: $A = bh/2$
2. Area of square, given the side length s: $A = s^2$
3. Area of square, given the diagonal length d: $A = d^2/2$
4. Area of a rhombus, given the length of the diagonals d_1 and d_2: $A = d_1 d_2/2$
5. Area of a circle given the radius r: $A = \pi r^2$
6. Area of a trapezoid given bases b_1, b_2, and the height h: $A = h(b_1 + b_2)/2$
7. Volume of cylinder or prism given base area B and height h: $V = Bh$
8. Volume of cone or pyramid given base area B and height h: $V = Bh/3$
9. Volume of sphere given radius r: $V = 4\pi r^3/3$
10. Surface area of sphere given radius r: $A = 4\pi r^2$

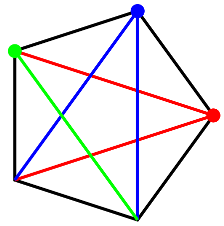
The number of diagonals in a regular polygon

A diagonal of a polygon is a line segment joining any two nonadjacent vertices. Let us count the number of diagonals in a few different regular polygons. We will use n to denote the number of sides in the polygon and N_n to denote the number of diagonals:

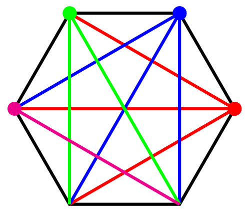
- The square ($n = 4$): $N_4 = 1 + 1 = 2$



- The pentagon ($n = 5$): $N_5 = 2 + 2 + 1 = 5$

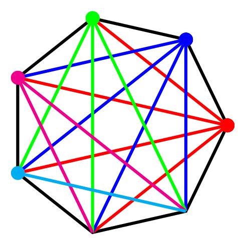


- The hexagon ($n = 6$): $N_6 = 3 + 3 + 2 + 1 = 9$



The number of diagonals in a regular polygon (Cont.)

- The heptagon ($n = 7$): $N_7 = 4 + 4 + 3 + 2 + 1 = 14$



- The general n-sided polygon:

$$\begin{aligned} N_n &= (n-3) + (n-3) + (n-4) + (n-5) + \cdots + 1 \\ &= (n-3) + T_{n-3} = n - 3 + \frac{(n-3)(n-2)}{2} \\ &= (n-3)\left[1 + \frac{(n-2)}{2}\right] = (n-3)\left[\frac{2}{2} + \frac{(n-2)}{2}\right] \\ &= (n-3)\frac{2+n-2}{2} \\ &= \boxed{\frac{n(n-3)}{2}} \end{aligned}$$

Example Problem Solved Using Number of Diagonals

Problem Suppose you are a gym teacher with 35 students in your ping-pong class. You need to set up a round-robin tournament in which every student plays a match against every other student. How many matches should be scheduled?

Solution Consider placing each student at the vertex of a 35-sided regular polygon. Asking for the number of matches is equivalent to asking the number of diagonals plus the number of sides of the polygon. We have

$$N_{35} = \frac{35(35-3)}{2} = \frac{35 \times 32}{2} = 560.$$

To this number we must add the number of sides on the polygon:

$$\# \text{ Matches} = 560 + 35 = 595$$