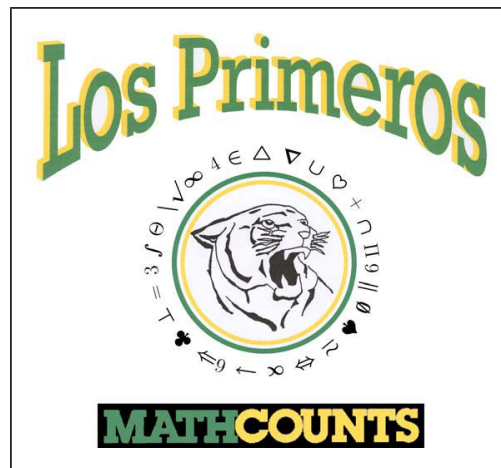


# Arithmetic, Algebra, Number Theory

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## Types of Numbers

- ▶ **Natural Numbers** The counting numbers:  $1, 2, 3, \dots$
- ▶ **Prime Number** A natural number with exactly two factors: itself and 1. Examples: 2, 3, 5, 7, 11,  $\dots$  Numbers that are not prime are **composite**.
- ▶ **Integers**  $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
- ▶ **Rational Numbers** can be written as  $p/q$ , where  $p$  is an integer and  $q$  is a nonzero integer. Examples: 1.31,  $7/8$ ,  $-3.1415$ ,  $0.9\overline{9}$ . **All rational numbers have either a finite or a repeating decimal representation.**
- ▶ **Irrational Numbers** Real numbers that can not be expressed as a ratio of integers. Equivalently, real numbers that have neither a finite nor repeating decimal representation. Examples:  $\pi$ ,  $\sqrt{2}$ ,  $3.101001000100001\dots$
- ▶ **Real Numbers** The set of numbers containing both the rationals and irrationals.

## Rules for Exponents and Radicals

Rule	Example
$x^{-a} = \frac{1}{x^a}$	$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$
$(x^a)^b = x^{ab} = (x^b)^a$	$(10^3)^2 = 10^{3 \times 2} = 10^6$
$x^a x^b = x^{a+b}$	$10^2 10^3 = 10^5$
$\frac{x^a}{x^b} = x^{a-b}$	$\frac{10^4}{10^3} = \frac{10000}{1000} = 10^1 = 10$
$x^a y^a = (xy)^a$	$5^2 6^2 = (5)(5)(6)(6) = (5 \times 6)^2$
$x^{\frac{1}{n}} = \sqrt[n]{x}$	$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 4$



### Exponent Practice Problems

(Answers should be simplified as much as possible)

1. Simplify  $\sqrt{75} = \sqrt{5^2 \times 3} = \sqrt{5^2} \times \sqrt{3} = 5\sqrt{3}$ .
2. Which of the following numbers are rational?  $\boxed{-25.6}$ ,  $\boxed{93/2}$ ,  $3\sqrt{-5}$ ,  $\boxed{70}$ ,  $\boxed{\sqrt{121} = 11}$ ,  $\boxed{33/7}$ ,  $\sqrt{122}$
3. Simplify  $(8 \times 27)^{\frac{2}{3}} = \left(2^3 \times 3^3\right)^{\frac{2}{3}} = \left(6^3\right)^{\frac{2}{3}} = \left(6^{3/3}\right)^2 = 36$
4. Simplify  $(25 \times 125)^{\frac{1}{5}} = \left(5^2 \times 5^3\right)^{\frac{1}{5}} = \left(5^5\right)^{\frac{1}{5}} = 5$



## Generating or Testing for Prime Numbers

**Trial Division** To see if an individual (small) integer is prime, trial division works well: just divide by all the primes less than (or equal to) its square root. For example, to show 211 is prime, just divide by 2, 3, 5, 7, 11, and 13. Since none of these divides the number evenly, it is a prime.

**Sieve of Eratosthenes** If we wish to generate a list of consecutive prime numbers less than some given number, the elimination technique called the sieve of Eratosthenes is faster.



### Prime Numbers—The Sieve of Eratosthenes

Recall that the prime numbers are natural numbers having exactly two factors (the number itself and 1). The most efficient way to find all of the small primes (say all those less than 10,000,000) is by using the **Sieve of Eratosthenes** (ca 240 BC):

1. Start with a list of natural numbers from 2 to, say, 37.

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24  
25 26 27 28 29 30 31 32 33 34 35 36 37

2. The first number 2 is prime. Retain it, but cross out all multiples of 2 in the list.

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~ 13 ~~14~~ 15 ~~16~~ 17 ~~18~~ 19 ~~20~~ 21 ~~22~~ 23  
~~24~~ 25 ~~26~~ 27 ~~28~~ 29 ~~30~~ 31 ~~32~~ 33 ~~34~~ 35 ~~36~~ 37

3. The first remaining number 3 is prime. Retain it, but cross out all multiples of 3 in the list.

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ 23  
~~24~~ 25 ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~ 31 ~~32~~ ~~33~~ ~~34~~ 35 ~~36~~ 37



## Prime Numbers—The Sieve of Eratosthenes (Cont.)

4. The first remaining number 5 is prime. Retain it, but cross out all multiples of 5 in the list.

2 3 5 7 11 13 17 19 23 29 31 37

5. The first remaining number 7 is prime. Retain it, but cross out all multiples of 7 in the list.

2 3 5 7 11 13 17 19 23 29 31 37

6. The first remaining number is  $11 > \sqrt{37}$  so we do not need to check for multiples of it or any of the other remaining numbers in the list (why?). All the remaining numbers in the list are primes: 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37



## GCF: Greatest Common Factor

The greatest common factor (GCF) of two natural numbers is the greatest factor that divides both of the numbers.

**Examples:**  $\text{GCF}(2, 3) = 1$ ,  $\text{GCF}(2, 4) = 4$ ,  $\text{GCF}(10, 15) = 5$ .

To find the GCF of two numbers:

1. List the prime factors of each number. Include all prime factors in both lists (may require a zero exponent).
2. Multiply each factor the lesser number of times that it occurs in either factor.

▶ **Example:** Find  $\text{GCF}(90, 24)$

▶ Prime factors of 90:  $2^1 \times 3^2 \times 5^1$

▶ Prime factors of 24:  $2^3 \times 3^1 \times 5^0$

▶ Therefore,  $\text{GCF}(90, 24) = 2^1 \times 3^1 \times 5^0 = 6$ .



## LCM: Least Common Multiple

A **common multiple** is a number that is a multiple of two or more numbers. The common multiples of 3 and 4 are 0, 12, 24, . . . The **least common multiple** (LCM) of two numbers is the smallest number (not zero) that is a multiple of both. So  $\text{LCM}(3, 4) = 12$ . To find the LCM of two numbers:

1. List the prime factors of each number. Include all prime factors in both lists (may require a zero exponent).
2. Multiply each factor the greater number of times that it occurs in either factor.

- ▶ **Example:** Find  $\text{LCM}(90, 24)$
- ▶ Prime factors of 90:  $2^1 \times 3^2 \times 5^1$
- ▶ Prime factors of 24:  $2^3 \times 3^1 \times 5^0$
- ▶ Therefore,  $\text{LCM}(90, 24) = 2^3 \times 3^2 \times 5^1 = 360$ .



## Product of GCF and LCM

Note that

$$\text{GCF}(90, 24) \times \text{LCM}(90, 24) = 6 \times 360 = 2160 = 90 \times 24$$

This is true in general:

$$\boxed{\text{GCF}(a, b) \times \text{LCM}(a, b) = a \times b}$$

The product of the **GCF** and **LCM** of two natural numbers is equal to the product of the numbers themselves.



## Repeating Decimals

**Example:**  $x = 0.32\overline{49} \equiv 0.32494949\dots$

To convert the repeating decimal form to an equivalent fraction:

1. Multiply by a power of 10 to put the decimal point just before the repeating part:

$$100x = 32.\overline{49}$$

2. Multiply by a power of 10 that matches the length of the repeating part:

$$100 \times 100x = 10000x = 3249.\overline{49}$$

3. Subtract the two equations obtained above:

$$10000x - 100x = 3249.\overline{49} - 32.\overline{49}$$

$$9900x = 3217$$

$$x = \frac{3217}{9900}$$



## Some Common Repeating Decimals

$$0.\overline{1} = \frac{1}{9}$$

$$0.\overline{6} = \frac{6}{9} = \frac{2}{3}$$

$$0.\overline{2} = \frac{2}{9}$$

$$0.\overline{7} = \frac{7}{9}$$

$$0.\overline{3} = \frac{3}{9} = \frac{1}{3}$$

$$0.\overline{8} = \frac{8}{9}$$

$$0.\overline{4} = \frac{4}{9}$$

$$0.\overline{9} = \frac{9}{9} = 1$$

$$0.\overline{5} = \frac{5}{9}$$



## Counting the Number of Factors

### ▶ Prime Numbers

$$7 = 1 \times 7 \quad (2 \text{ factors})$$

$$11 = 1 \times 11 \quad (2 \text{ factors})$$

Prime numbers have exactly 2 factors

### ▶ Prime Number Raised to Integer Power

$$9 = 3^2 \text{ has 3 factors: } 1, 3, 9$$

$$27 = 3^3 \text{ has 4 factors: } 1, 3, 9, 27$$

A prime raised to a positive integer power  $n$  has  $n + 1$  factors.

### ▶ Product of Two Distinct Primes

$$21 = 3 \times 7, \quad 4 \text{ factors: } 1, 3, 7, 21$$

A product of two distinct primes has exactly 4 factors



## Counting the Number of Factors (Cont.)

### ▶ Product of Prime Numbers Raised to Powers

$$100 = 4 \times 25 = 2^2 \times 5^2 \text{ has 9 factors: } 1, 2, 4, 5, 10, 20, 25, 50, 100$$

$$\text{Note that } 9 = 3 \times 3 = (2 + 1) \times (2 + 1)$$

$$72 = 8 \times 9 = 2^3 \times 3^2 \text{ has 12 factors: } 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72$$

$$\text{Note that } 12 = 4 \times 3 = (3 + 1) \times (2 + 1)$$

- ▶ Let  $P_1, P_2, P_3, \dots$  be prime numbers and let  $n_1, n_2, n_3, \dots$  be natural numbers. Then the number of factors of

$$P_1^{n_1} \times P_2^{n_2} \times P_3^{n_3} \times \dots$$

is

$$(n_1 + 1) \times (n_2 + 1) \times (n_3 + 1) \times \dots$$



## Counting the Number of Factors (Cont.)

▶ **Example: How many factors does 150 have?**

1. Write the number as the product of primes raised to powers:

$$150 = 6 \times 25 = 2 \times 3 \times 5^2 = 2^1 \times 3^1 \times 5^2$$

2. Add one to each exponent and then multiply them:

$$\# \text{ Factors} = (1 + 1) \times (1 + 1) \times (2 + 1) = 2 \times 2 \times 3 = 12.$$

▶ **Practice Problems:** Find the # of factors of each number:

$$22 = 2^1 \implies (1 + 1) = 2 \text{ factors}$$

$$88 = 2^3 \implies (3 + 1) = 4 \text{ factors}$$

$$1010 = 2^1 \times 5^1 \implies (1 + 1)(1 + 1) = 2 \times 2 = 4 \text{ factors}$$

$$2020 = 2^2 \times 5^1 \implies (2 + 1)(1 + 1) = 3 \times 2 = 6 \text{ factors}$$

$$200200 = 2^3 \times 5^2 \implies (3 + 1)(2 + 1) = 4 \times 3 = 12 \text{ factors}$$

