Name:_____

Los Primeros MATHCOUNTS 2004–2005
Homework 4
May 26, 2004

1. A farmer has 100 feet of fence with which to fence off his garden. He wishes to enclose the largest area possible for his garden and he doesn't know what shape to choose for his plot. Should he pick an equilateral triangle, square, or circle? To help him decide, calculate the area of each to the nearest tenth of a square foot.

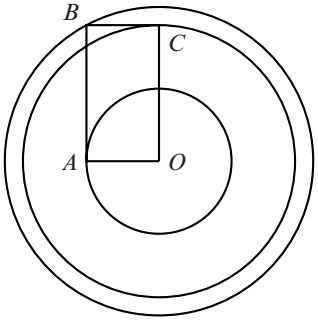
2. A rectangular dog house measures 6 feet by 2 feet. The dog is tied to one corner of the doghouse with a 4 foot leash. What is the size of the area in square feet that the dog can sweep out with his fully extended leash? Express your answer as a number multiplied by π. (Hint: Draw a picture)

3. Three concentric circles have their centers at O, and $OABC$ is a rectangle. Point A lies on the inner circle, Point B lies on the outer circle, and Point C lies on the middle circle. If the radius of the inner circle is 3, find

   (a) The area of the inner circle.

   (b) The area of the region between the two outer circles (such a region bounded by concentric circles is called an *annulus*). (Hint: Use what you know about rectangles and Pythagoras)

   (c) Can you find the area of the middle circle?



4. What is the area of a circle if a triangle with side lengths 80, 82, and 18 is inscribed in it?

Los Primeros MATHCOUNTS 2004–2005
Answer Key for Homework 4
May 26, 2004

1. A farmer has 100 feet of fence with which to fence off his garden. He wishes to enclose the largest area possible for his garden and he doesn't know what shape to choose for his plot. Should he pick an equilateral triangle, square, or circle? To help him decide, calculate the area of each to the nearest tenth of a square foot.

   Answer:

   (a) Equlateral triangle: The side length is $b = 100/3$ and the height from any vertex to the opposite side is $h = \frac{\sqrt{3}}{2}b = 50\sqrt{3}/3$ so that the area of the triangle is

   $$\text{Triangle area} = \frac{1}{2}bh = \frac{100 \times 50\sqrt{3}}{2 \times 3 \times 3} \approx 481.1 \text{ square feet.}$$

   (b) Square: The side length of the square is $100/4 = 25$ and the area of the square is

   $$\text{Square area} = 25 \times 25 = 625.0 \text{ square feet.}$$

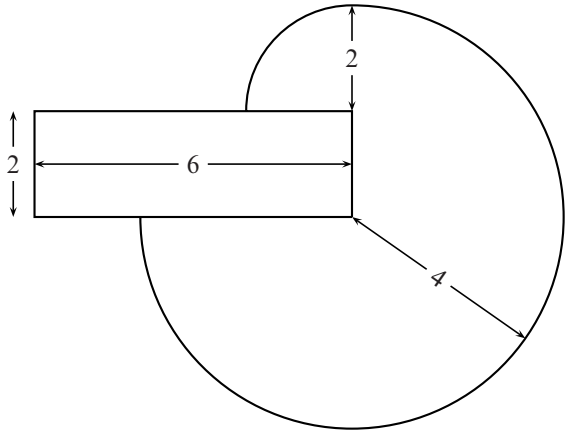
   (c) Circle: The radius of the circle is $r = C/(2\pi) = 100/(2\pi)$ and the area is

   $$\text{Circle area} = \pi r^2 = \pi \frac{10000}{4\pi^2} = \frac{2500}{\pi} \approx 795.8 \text{ square feet.}$$

   Therefore the circle is by far the best choice. In fact, it can be shown using the methods of the calculus of variations that the circle is the best choice of any planar figure!

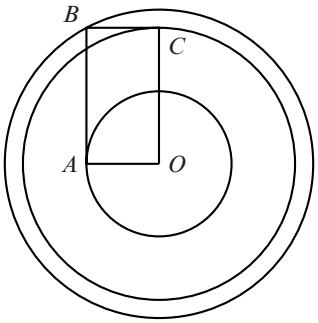
2. A rectangular dog house measures 6 feet by 2 feet. The dog is tied to one corner of the doghouse with a 4 foot leash. What is the size of the area in square feet that the dog can sweep out with his fully extended leash? Express your answer as a number multiplied by π. (Hint: Draw a picture)

   Answer: Consider tying the leash to the lower right corner of the house shown in the diagram. Until the leash is pointed straight up, it is sweeping out a circular sector of radius 4 centered on the bottom right corner. Once it passes the upper right corner, it will sweep out a circular arc of radius $4 - 2 = 2$ centered on the upper right corner. Thus, the total area is equal to three fourths of the area of a circle of radius 4 feet added to one fourth of the area of a a circle of radius 2 feet. Performing this calculation gives



$$\text{Area} = \frac{3}{4}\pi 4^2 + \frac{1}{4}\pi 2^2 = 12\pi + \pi = 13\pi$$

3. Three concentric circles have their centers at O, and $OABC$ is a rectangle. Point A lies on the inner circle, Point B lies on the outer circle, and Point C lies on the middle circle. If the radius of the inner circle is 3, find



   (a) The area of the inner circle.

   (b) The area of the region between the two outer circles (such a region bounded by concentric circles is called an *annulus*). (Hint: Use what you know about rectangles and Pythagoras)

   (c) Can you find the area of the middle circle?

Answer: We will label the three circles as 1, 2, and 3, starting from the innermost.

(a) Since we are given that $OA = 3$ then the small circle area is $A_1 = \pi(OA)^2 = 9\pi$.

(b) By the Pythagorean Theorem, $(OA)^2 + (OC)^2 = (OB)^2$ so that $(OB)^2 - (OC)^2 = (OA)^2 = 9$. But the area of the annulus is the difference of the areas of circles 3 and 2, so that

$$\text{Annulus area} = \pi(OC)^2 - \pi(OB)^2 = \pi\left[(OB)^2 - (OC)^2\right] = 9\pi$$

which is the same as the area of the small circle!

(c) No, it is not possible with the information given.

4. What is the area of a circle if a triangle with side lengths 80, 82, and 18 is inscribed in it?

Answer: Note that $18^2 + 80^2 = 6724 = 82^2$ so that the triangle must be a right triangle. According to a theorem proved in class, the hypotenuse of a right triangle inscribed in a circle is a diameter of that circle, thus the radius of the circle is $r = 82/2 = 41$ and the area is $\pi r^2 = 1681\pi$.