Los Primeros MATHCOUNTS 2004–2005
Introduction to Combinatorics
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What is Combinatorics?

Combinatorics is a fancy word for counting. Combinatorics is concerned with determining the the number of logical possibilities of some event without necessarily listing all the particular outcomes. One can often perform calculations involving probabilities simply by counting the possible outcomes. Combinatorics often requires counting the number of possible permutations (rearrangements) or combinations (groupings) of a set of objects.

Fundamental Principle of Counting

If some event can in occur in n_1 different ways, and if following this event, a second event can occur in n_2 different ways, and if following this second event, a third event can occur in n_3 ways, . . ., then the number of ways the events can occur in the order indicated is $n_1 \times n_2 \times n_3 \ldots$.

Example (License Plates)

Suppose a license plate contains three letters (all capitals) followed by two digits with the first digit not zero. How many different license plates can be printed?

Each letter can be printed in twenty-six different ways, the first digit in nine ways, and the last digit in ten ways. Therefore, there are

$$26 \times 26 \times 26 \times 9 \times 10 = 1{,}581{,}840$$

different plates that can be printed.

Fundamental Principle of Counting (Cont.)

Example (Base 2 Counting)

How many distinct numbers can be represented by an 8-bit binary number?

In a binary number, each binary digit (bit) is either 0 or 1: there are exactly two choices for each bit. Therefore, the number of unique bit combinations is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$$

An 8-bit binary number is called a byte. This is the fundamental unit of memory storage in modern computers. We see that if one byte is used to represent a letter of the alphabet, then the largest alphabet that could be represented this way would consist of 256 letters. Although this is plenty for English, providing enough room for both uppercase and lowercase letters and the decimal digits, it is far too few for many other languages, such as Chinese, which contains thousands of characters.

Fundamental Principle of Counting (Cont.)

Example (Base 8 Counting)

How many distinct numbers can be represented by a 6-digit octal number?

In an octal number, each digit is one of numbers $0, 1, 2, \ldots, 7$: there are exactly eight choices for each digit. Therefore, the number of unique bit combinations is

$$8 \times 8 \times 8 \times 8 \times 8 \times 8 = 8^6 = 262{,}144$$

Fundamental Principle of Counting (Cont.)

Example (Counting without Duplicate Digits)

How many different numbers can be generated using six digits, if none of the digits are repeated?

There are 10 choices for the first digit. Once this digit has been selected, there remain 9 choices for the second digit. Once the first two digits have been chosen, there remain 8 choices for the third digit, and so on. Using the Fundamental Principle of Counting, we see that there are

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151{,}200$$

different numbers that can be generated in this way.

Factorial Notation

The product of the positive integers from 1 to n inclusive is denoted by $n!$ (read as "n factorial"):

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n.$$

It is convenient to define $0! = 1$. As we will see, this convention makes many formulas come out "nicer" for the boundary cases.
Example: $3! = 3 \times 2 \times 1 = 1 \times 2 \times 3 = 6$.
Example: $4! = 4 \times 3 \times 2 \times 1 = 24$.
Example: Express $8 \times 7 \times 6$ in factorial notation.

$$8 \times 7 \times 6 = 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

Example: Compute $\frac{25!}{22!}$.

$$\frac{25!}{22!} = \frac{25 \times 24 \times 23 \times 22!}{22!} = 25 \times 24 \times 23 = 13,800.$$

Factorial Example Problems

(Students should copy these answers into their notes)

1. $0! =$
2. $1! =$
3. $2! =$
4. $5! =$
5. Express $9 \times 8 \times 7$ in factorial notation:
6. Evaluate $\frac{20!}{18!}$:

Permutations

Any particular arrangement of a set of n objects in a given order is called a permutation of these objects (taken all at a time). Any arrangement of $k \leq n$ of these objects in a given order is called a k-permutation or a permutation of the n objects taken k at a time.
For example, consider the set of letters $a, b, c,$ and d. Then cbd is a permutation of the four letters taken three at a time.
The number of permutations of n objects taken k at a time is denoted by

$$P(n, k), \quad {}_nP_k, \quad P_{n,k}, \quad P_k^n, \quad \text{or} \quad (n)_k.$$

We will use only ${}_nP_k$.

Permutations Example: Eight Objects Five at a Time

How many permutations are there for eight objects, say a, b, c, d, e, f, g, h, taken five at a time? Let the general five-letter word be represented by the following five boxes:
$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$. Now the first letter can be chosen in eight different ways; following this, the second letter can be chosen in seven different ways; following this, the third letter can be chosen in six different ways; and so on. We write each number in its appropriate box as follows: $\boxed{8}\ \boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4}$. Thus by the fundamental principle of counting there are $8 \times 7 \times 6 \times 5 \times 4 = 6720$ possible five-letter words without repetitions from the eight letters. In other words, there are 6720 permutations of eight objects taken five at a time:

$${}_8P_5 = 6720 = 8 \times 7 \times 6 \times 5 \times 4 = \frac{8!}{3!} = \frac{8!}{(8-5)!}$$

The General Formula for ${}_nP_k$

The general formula is obtained by reasoning identical to that of the previous example. Thus,

$$\begin{aligned}
{}_nP_k &= \underbrace{(n-0)(n-1)(n-2)\cdots(n-[k-1])}_{k \text{ factors}} \\
&= n(n-1)(n-2)\cdots(n-k+1) \\
&= n(n-1)(n-2)\cdots(n-k+1)\tfrac{(n-k)(n-k-1)(n-k-2)\cdots(3)(2)(1)}{(n-k)(n-k-1)(n-k-2)\cdots(3)(2)(1)} \\
&= \boxed{\frac{n!}{(n-k)!}}
\end{aligned}$$

Permutations versus Combinations

Sometimes we need to count the ways that a group of objects can be arranged into sets without regard to order. For instance, suppose we wish to count the number of ping-pong matches needed for each student in a class to play each other student. In this case, counting the number of permutations of the class taken 2 at a time is not appropriate. The reason is that permutations recognize order. That is, both (Jack,Jill) and (Jill,Jack) would counte as separate permutations, when they both represent a single ping-pong match between Jack and Jill. We shouldn't have count both of them. We need a way of counting that doesn't distinguish based on reorderings.

Such a way is to count combinations rather than permutations.

Combinations

Suppose we have a collection of n objects. A combination of these n objects taken k at a time is any set containing k of the objects, where we do not distinguish combinations based on order.
Combinations differ from permutations in that the order of the objects matters in permutations and does not matter in combinations.
For example, the combinations of the letters a, b, c, d taken three at a time are

$$\{a, b, c\}, \ \{a, b, d\}, \ \{a, c, d\}, \ \{b, c, d\}, \ \text{or simply} \quad abc, \ abd, \ acd, \ bcd.$$

It is important to understand that the following combinations are equal:

$$abc, \ acb, \ bac, \ bca, \ cab, \ cba$$

Each denotes the same set $\{a, b, c\}$. Question: Could you have predicted that there are six arrangements of these three letters?

Combinations versus Permutations

(Students should fill in the answers in their notes)

In the following, would you count permutations or combinations?

1. The number of three-person committees that can be chosen from a class of 30 students.
2. The number of three-letter "words" (including nonsense words) that can be formed from the letters "t", "a", and "c".
3. The number of routes a mailman can take to deliver letters to his patrons.
4. The number of different starting lineups that can be selected from a basketball team of 12 players.
5. The number of ways to arrange the 12 songs on a CD.

Notation for the Number of Combinations of *n* Things Taken *k* at a Time

Several notations in common use are

$$C(n,k) = \binom{n}{k} = C_{n,k} = {}_nC_k = C^n_k$$

with $\binom{n}{k}$ being the most popular. It is read as "n choose k" or "n things k at a time" or "the number of combinations of n things taken k at a time." In Mathcounts We will use the notation ${}_nC_k$ most of the time.

Example: ${}_4C_3$

How many combinations are there of four objects taken three at a time? Let's label the four objects a, b, c, and d. Each combination consisting of three objects determines $3! = 3 \times 2 \times 1 = 6$ permutations of the objects in the combination:

Combinations	Permutations
abc	*abc*, *acb*, *bac*, *bca*, *cab*, *cba*
adb	*adb*, *abd*, *dab*, *dba*, *bad*, *bda*
acd	*acd*, *adc*, *cad*, *cda*, *dac*, *dca*
bcd	*bcd*, *bdc*, *cbd*, *cdb*, *dbc*, *dcb*

Thus the number of combinations multiplied by 3! equals the number of permutations:

$$ {}_4C_3 \times 3! = {}_4P_3 \qquad \text{or} \qquad {}_4C_3 = \frac{{}_4P_3}{3!}.$$

But, ${}_4P_3 = 4 \times 3 \times 2 = 24$ and $3! = 6$, so ${}_4C_3 = 24/6 = 4$ as we can observe from the table.

General Formula for ${}_nC_k$

Since any combination of n objects taken k at a time determines $k!$ permutations of the objects in the combination, we can conclude that

$$ {}_nP_k = k!\, {}_nC_k $$

or, finally

$$ \boxed{{}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!}} $$

How to Compute ${}_nC_k$

Recall that there are k factors in

$$ {}_nP_k = \frac{n!}{(n-k)!} = \underbrace{n(n-1)(n-2)\cdots(n-k+1)}_{k \text{ factors}} $$

and that there are also k factors in

$$ k! = \underbrace{k(k-1)(k-2)\cdots 1}_{k \text{ factors}} $$

so that

$$ {}_nC_k = \frac{n!}{k!(n-k)!} = \underbrace{\frac{n}{k} \times \frac{n-1}{k-1} \times \frac{n-2}{k-2} \times \cdots \times \frac{n-k+1}{1}}_{k \text{ factors}} $$

Using this formula for ${}_nC_k$ will help to avoid overflows on your calculator!

Practice Problem 1

Burger Shack offers only one type of hamburger but seven different toppings. How many types of burgers can be created by choosing any four toppings? Ignore the order that the four toppings are applied.

Practice Problem 1 Solution

Burger Shack offers only one type of hamburger but seven different toppings. How many ways types of burgers can be created by choosing any four toppings?

Reasoning Assume first that the order of the toppings matters. We have 7 choices for the first topping, 6 choices left for the second, 5 choices left for the third, and 4 choices left for the fourth, giving a total of $7 \times 6 \times 5 \times 4$ different arrangements of the four toppings. However, for each set of four toppings selected, we counted each reordering (permutation) of these four as a separate case. Thus we overcounted by a factor equal to the number of rearrangements (permutations) of four objects, or 4!. Thus, the number of different burgers we seek is

$$\frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35$$

Practice Problem 1 Solution, (Cont.)

Burger Shack offers only one type of hamburger but seven different toppings. How many ways types of burgers can be created by choosing any four toppings?

Using the Formula Since the order of the toppings doesn't matter, we want to count the number of combinations of 7 things taken 4 at a time:

$$_7C_4 = \frac{7!}{4!(7-4)!} = \frac{7}{4} \times \frac{6}{3} \times \frac{5}{2} \times \frac{4}{1} = 7 \times 5 = 35.$$

Practice Problem 2

How many different ways can Jenny arrange her schedule of 6 classes?

Reasoning Jenny has 6 choices for her first class. She then has 5 remaining choices for her second class, 4 for her third class, and so on. Therefore, there are $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different ways to arrange her schedule.

Using the Formula We want to know the number of permutations of 6 things taken 6 at a time, or

$$_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6!}{1} = 6! = 720.$$

Permutations and Repetitions

Permutations of "DADDY" Suppose we want to find the number of five-letter words that can be created by rearranging the letters in the word "DADDY". First, consider the number of permutations of the symbols $D_1AD_2D_3Y$, where we will temporarily distinguish between the three D's. Clearly there are $_5P_5 = 5!$ such permutations. However, we note that the following six permutations

$D_1D_2D_3AY$, $D_1D_3D_2AY$, $D_2D_1D_3AY$, $D_2D_3D_1AY$, $D_3D_1D_2AY$, and $D_3D_2D_1AY$

all produce the same word when the subscripts are removed. The 6 comes from the fact that there are $_3P_3 = 3! = 6$ permutations for placing the three D's in the first three positions of this permutation. This will be true for any choice of placement of the three D's. Thus there are

$$\frac{5!}{3!} = \frac{120}{6} = 20$$

different five-letter words obtainable by rearranging the word DADDY.

Permutations of "DADDA"

How many five-letter words can be formed by rearranging the letters in the word "DADDA"? As before, we note that if all the letters were distinguishable, then we would have $_5P_5 = 5!$ possible rearrangements. However, three D's are identical and two A's are identical in this word. By similar reasoning to the previous example, we see that we have overcounted by the product of the number of rearrangements of three things times the number of possible rearrangements of two things $3! \times 2!$. Therefore, the number of five-letter words obtainable by rearranging the letters in "DADDA" is

$$\frac{5!}{3!\,2!} = \frac{120}{6 \times 2} = 10.$$

General Formula for Permutations with Repetitions

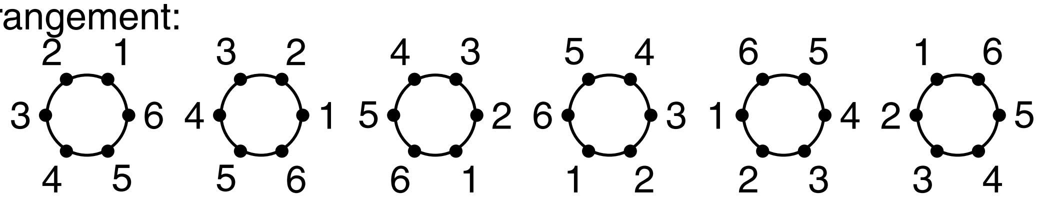
The number of permutations of n objects of which n_1 are alike, n_2 are alike, . . ., n_r are alike is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Circular Permutations

Question: How many ways can six people be seated at a round table?

When attacking this type of problem you must first understand that we do not consider two different seatings to be distinct if one can be obtained from the other by simply rotating all the people around the table, while maintaining their relative positions. For example, if the people are numbered 1–6, the following seatings are all considered to be equivalent and count as only one seating arrangement:



because each person has the same neighbors to his left and right in each case.

We see that we can fix the location of one of the diners, say #6, and then there are 5 remaining slots to fill, which can be done in ${}_5P_5 = 5! = \boxed{120}$ ways. **In general, there are (n – 1)! ways to arrange n things around a circle.**

Pascal's Triangle and Powers of 11

Let's look at the first few integer powers of 11:

$$11^0 = 1$$
$$11^1 = 1 \quad 1$$
$$11^2 = 1 \quad 2 \quad 1$$
$$11^3 = 1 \quad 3 \quad 3 \quad 1$$
$$11^4 = 1 \quad 4 \quad 6 \quad 4 \quad 1$$

The decimal digits in these powers of 11 are equal to the following combinations of n things k at a time:

$$ {}_0C_0 $$
$$ {}_1C_0 \quad {}_1C_1 $$
$$ {}_2C_0 \quad {}_2C_1 \quad {}_2C_2 $$
$$ {}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 $$
$$ {}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4 $$

What is the connection between Pascal's triangle (consisting of ${}_nC_k$, the number of combinations of n things taken k at a time) and the digits occuring in the first few integral powers of 11?

Binomial Theorem and Pascal's Triangle

The connection between Pascal's triangle and the digits occurring in the first few integer powers of 11 is given by a mathematical identity known as the binomial theorem. Pascal's Triangle is named after the French mathematician, Blaise Pascal (1623–1662), although it had been described 5 centuries earlier by the Chinese and also and the by the Persians. The binomial theorem was known for the case $n = 2$ by Euclid ca 300 BC, and stated in modern form by Pascal.

A binomial is just the sum of two numbers: $a + b$.

Binomials Raised to Powers

Consider the binomial

$$P = (a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b).$$

We want to keep track of where the a's and b's in the product come from, so we will temporarily add subscripts to each factor in the product:

$$P = (a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)(a_5+b_5)$$

If we multiply this out, we get terms like $a_1 b_2 a_3 a_4 b_5$. The total number of terms will be 2^5 (why?). These terms result from selecting one of the terms (a_k or b_k) from each factor in the original product. Note that the number of as and bs in each term must add up to 5. How many terms are there with three as and two bs? There must be ${}_5C_3 = {}_5C_2 = \frac{5!}{2!\,3!} = 10$ (why?)

The Binomial Theorem

Since there really aren't any subscripts on the as and bs, each of these 10 terms is identical and is equal to a^3b^2. Since there are 10 of them, the product will contain the term $10a^3b^2$.

From these considerations, one can prove that for any numbers a and b and any nonnegative integer n:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} {}_nC_k a^{n-k} b^k \\ &= {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \cdots \\ &\qquad \cdots + {}_nC_{n-1} a^1 b^{n-1} + {}_nC_n a^0 b^n.\end{aligned}$$

This is the motivation for referring to the quantity ${}_nC_k$ as a binomial coefficient.

Examples for Binomial Theorem

Example ($n = 2$)

$$(a+b)^2 = \sum_{k=0}^{2} {}_2C_k a^{2-k} b^k = {}_2C_0 a^2 b^0 + {}_2C_1 a^1 b^1 + {}_2C_2 a^0 b^2 = a^2 + 2ab + b^2$$

Example ($n = 2$ with $a = 5$ and $b = 2$)

$$(5+2)^2 = \sum_{k=0}^{2} {}_2C_k 5^{2-k} 2^k = {}_2C_0 5^2 2^0 + {}_2C_1 5^1 2^1 + {}_2C_2 5^0 2^2$$
$$= 5^2 + 2 \times 5 \times 2 + 2^2 = 49$$

Binomial Theorem and Pascal's Triangle (Cont.)

The theorem is sometimes very convenient when it is easy to raise a and b to various powers and more difficult to calculate $(a + b)$ raised to the same powers. For example $11 = 10 + 1$ and it is very easy to raise 10 and 1 to any integer powers.

Example

$$\begin{aligned} 11^3 &= (10+1)^3 \\ &= {}_3C_0 10^3 1^0 + {}_3C_1 10^2 1^1 + {}_3C_2 10^1 1^2 + {}_3C_3 10^0 1^3 \\ &= {}_3C_0 \times 1000 + {}_3C_1 \times 100 + {}_3C_2 \times 10 + {}_3C_3 \times 1 \\ &= 1 \times 1000 + 3 \times 100 + 3 \times 10 + 1 \times 1 = 1331. \end{aligned}$$

Note that we can find all of the binomial coefficients for $n = 3$ by looking at the decimal digits in the number 11^3. Similarly, we can find all the binomial coefficients for $n = 2$ by examining the decimal representation of 11^2. But consider how multiplication by 11 is accomplished. . .

Binomial Theorem and Pascal's Triangle (Cont.)

Consider 11^2: $\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ +11 \\ \hline 121 \end{array}$ and 11^3: $\begin{array}{r} 121 \\ \times 11 \\ \hline 121 \\ +121 \\ \hline 1331 \end{array}$

Each successive power of 11 is obtained by adding two copies of the preceeding power, after shifting one of the factors one place to the left. This observation helps explain the workings of **Pascal's Triangle.**

Pascal's Triangle Begins with Decimal Expansion of $\mathbf{11^n}$

$$\begin{array}{rccccccccccccc}
11^0 = & & & & & & & 1 & & & & & & \\
11^1 = & & & & & & 1 & & 1 & & & & & \\
11^2 = & & & & & 1 & & 2 & & 1 & & & & \\
11^3 = & & & & 1 & & 3 & & 3 & & 1 & & & \\
11^4 = & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & & \dots & & & & & &
\end{array}$$

Pascal's Triangle

We can generalize the interpretation of Pascal's triangle to binomials involving any two numbers *a* and *b*.

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\
(a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\
&\cdots
\end{aligned}
$$

Pascal's Triangle (Cont.)

$$
\begin{aligned}
(a+b)^0 &= \binom{0}{0}\\
(a+b)^1 &= \binom{1}{0}a+\binom{1}{1}b\\
(a+b)^2 &= \binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2\\
(a+b)^3 &= \binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3\\
(a+b)^4 &= \binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4\\
(a+b)^5 &= \binom{5}{0}a^5+\binom{5}{1}a^4b+\binom{5}{2}a^3b^2+\binom{5}{3}a^2b^3+\binom{5}{4}ab^4+\binom{5}{5}b^5\\
(a+b)^6 &= \binom{6}{0}a^6+\binom{6}{1}a^5b+\binom{6}{2}a^4b^2+\binom{6}{3}a^3b^3+\binom{6}{4}a^2b^4+\binom{6}{5}ab^5+\binom{6}{6}b^6\\
&\cdots
\end{aligned}
$$

Pascal's Triangle (Cont.)

Retaining only the numeric coefficients we again obtain Pascal's Triangle:

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
                 ...
```

Pascal's triangle has the following interesting properties:

1. The first and last number in each row is 1 (since $\binom{n}{0} = \binom{n}{n} = 1$).
2. Every other number in the array can be obtained by adding the two numbers directly above it (since $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$).
3. The triangle has left/right mirror symmetry (since $\binom{n}{k} = \binom{n}{n-k}$).
4. The entries in each row sum to twice the value of the previous row. Row n sums to the value 2^n (counting rows starting from 0).

Pascal's Triangle Example Problem 1

Circle the coefficient corresponding to ${}_6C_3$ in Pascal's Triangle, below. Verify using the formula for combinations that the value shown is correct.

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
                 ...
```

Pascal's Triangle Example Problem 2

Calculate the following quantity without using pencil and paper or calculator:

$$99^3 + 3 \times 99^2 + 3 \times 99 + 1 = ?$$

Answer: 1,000,000. Note that

$$\begin{aligned} 99^3 + 3 \times 99^2 + 3 \times 99 + 1 &= 99^3 1^0 + 3 \times 99^2 1^1 + 3 \times 99^1 1^2 + 99^0 1^3 \\ &= (99 + 1)^3 \\ &= 100^3 = (10^2)^3 = 10^6. \end{aligned}$$