LPSS MATHCOUNTS 2004–2005

Lecture 1: Arithmetic Series—4/6/04

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Sequences and Series

Definition: A *sequence* is a list of numbers (in a particular order). The entries in the list are called *terms*.

Example: $1, 9, 2.5, 17$ is a sequence of four numbers and $17, 1, 9, 2.5$ is another (different) sequence.

Definition A *series* is just a sum consisting of two or more numbers.

Example: $1 + 9 + 2.5 + 17$ is a series.

Categorize the following as series or sequences:

$$1, 2, 3, 4, 5, 6 \tag{1}$$

$$5 + 10 + 15 + 20 \tag{2}$$

$$1 + 3 + 5 + 20 \tag{3}$$

$$4, 5, 6 \tag{4}$$

$$100 - 200 + 300 - 400 \tag{5}$$

Arithmetic Sequences and Series

Definition An *arithmetic* sequence is one where each number differs from its predecessor by a constant amount.

Examples: $1, 2, 3, 4, 5, \ldots, 100$ is an arithmetic sequence.

$1 + 2 + 3 + 4 + 5 + \cdot + 100$ is an arithmetic series.

$1 + 2 + 4 + 8$ is not an arithmetic series (it is a geometric series).

Classify the following series as arithmetic or not:

$$2 + 4 + 6 + 8 \tag{6}$$

$$1 + 2 + 4 + 6 + 8 \tag{7}$$

$$10 + 20 + 30 \tag{8}$$

$$40 + 45 + 50 + 60 \tag{9}$$

Counting the Number of Terms in an Arithmetic Series

Consider the following arithmetic series:

$$S = 1 + 2 + 3 + \cdots + 100$$

Clearly there are 100 terms. But if we subtract the first term from the last we get $100 - 1 = 99$. This is because there are 99 intervals and **there is always one more term than the number of intervals** (draw picture).

How many terms are there in the series: $T = 4 + 6 + 8 + \cdots + 80$?

The interval between terms is 2. The distance between the first and last term is $80 - 4 = 76$. The interval size is 2. The number of intervals is therefore

$$\frac{80 - 4}{2} = \frac{76}{2} = 38$$

and the number of terms is one more than the number of intervals or 39.

Counting the Number of Terms in an Arithmetic Series

Count the number of terms in the following arithmetic series:

$$20 + 40 + 60 + \cdots + 1020 \tag{10}$$

$$9 + 10 + 11 + 12 + \cdots + 1000 \tag{11}$$

$$-20 + (-15) + (-10) + \cdots + 45 \tag{12}$$

$$33 + 30 + 27 + \cdots + 6 \tag{13}$$

Carl Friedrich Gauss

- Born: 1777, Brunswick, Germany.
- Died: February 23, 1855, Göttingen, Germany
- One of the greatest mathematicians of all time.
- Fundamental contributions to many areas:
  - number theory
  - geometry
  - surface and line integrals
  - theory of complex variables
  - numerical analysis.
- Gauss displayed prodigious mathematical talent at a young age.
  - He could calculate before he could talk.

- By six years of age he was correcting his father’s wage calculations.
- At age eight, he astounded his teacher by instantly solving a busy-work problem: Sum all the integers from 1 to 100.
- At seventeen, he derived a construction for the regular 17-sided polygon using only straightedge and compass.

How Did Gauss Solve the Busy-work Problem?

Problem: Find the sum $S = 1 + 2 + 3 + \cdots + 100$.

Gauss realized that you could add this series to itself in reversed order and obtain a series whose terms are all equal:

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

Since there are 100 terms in the series, we have that $2S = 100 \times 101$ or $S = \frac{100\times101}{2} = 50 \times 101 = 5050$.

Another Way to Solve Gauss' Problem: Triangular Numbers

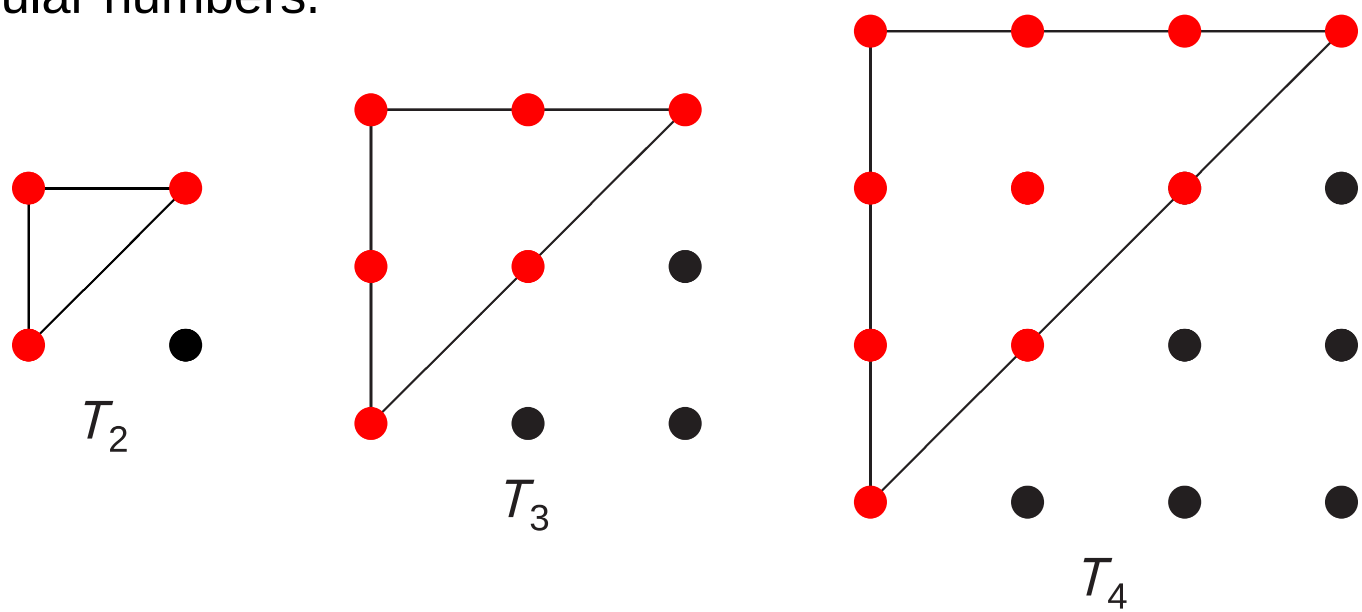
Let us generalize Gauss' problem a little bit. Let

$$\begin{aligned}
T_1 &= 1 \\
T_2 &= 1 + 2 = 3 \\
T_3 &= 1 + 2 + 3 = 6 \\
&\vdots \\
T_n &= 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},
\end{aligned}$$

so Gauss was asked to find the value of T_{100}. We call $\{T_1, T_2, \ldots\}$ the *triangular numbers*. The following diagrams show why:

Geometrical Interpretation of Triangular Numbers

Add up the number of red dots in each picture and you will obtain the triangular numbers.



Note that for each square we use roughly half the dots for our triangular number. Since there are n^2 total dots in a square of side length n, we should expect that $T_n \approx \frac{n^2}{2}$. In fact the general formula is easy to deduce:

$$T_n = \frac{n(n+1)}{2}$$

Derivation of Formula for T_n

- **Gauss' Method:**

$$\begin{array}{ccccccccccccc} T_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\ T_n & = & n & + & n-1 & + & n-2 & + & \cdots & + & 1 \\ \hline 2T_n & = & n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 \end{array}$$

Since there are n terms in the series, we have that $2T_n = n \times (n+1)$ or $T_n = \frac{n(n+1)}{2}$.

Derivation of Formula for T_n (cont.)

- **Geometrical Proof**

  Consider building a square with $n + 1$ dots on each side. Then there are $(n + 1)^2$ total dots, and there are $n + 1$ dots on the diagonal. Notice that there are T_n dots above the diagonal, which is equal to the number of dots below the diagonal. Then adding up the number of dots, we have $2T_n + (n + 1) = (n + 1)^2$ or

$$T_n = \frac{n(n + 1)}{2}.$$

Arithmetic Series

Recall the triangular numbers

$$T_n \equiv 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n = 1, 2, 3, \ldots$$

These are a special case of an *arithmetic series*:

$$S = \underbrace{a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n-1)d]}_{n \text{ terms}}.$$

Note that each term in an arithmetic series differs from its predecessor by a constant amount d.

Example of Arithmetic Series

Suppose an auditorium consists of 40 rows of seats. The first row contains 10 seats, the second row 12 seats, the third row 14 seats, and so on. Each row contains two more seats than its predecessor. How many seats S are there in the auditorium?

This is an arithmetic series:

$$
\begin{aligned}
S &= \underbrace{10 + 12 + 14 + 16 + \cdots + 88}_{\text{40 terms}} \\
&= 10 + (10 + 2) + (10 + 2 \times 2) + (10 + 3 \times 2) + \cdots + (10 + 39 \times 2) \\
&= \underbrace{a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n-1)d]}_{n \text{ terms}} .
\end{aligned}
$$

with $a = 10$, $d = 2$, and $n = 40$.

Arithmetic Series (Cont.)

We can find the number of seats S in the auditorium using the technique of Gauss:

$$
\begin{array}{ccccccccccc}
S & = & 10 & + & 12 & + & 14 & + & \cdots & + & 88 \\
S & = & 88 & + & 86 & + & 84 & + & \cdots & + & 10 \\
\hline
2S & = & 98 & + & 98 & + & 98 & + & \cdots & + & 98
\end{array}
$$

$$\text{Therefore: } S = \frac{40 \times 98}{2} = 20 \times 98 = 1960.$$

A similar calculation yields the formula for the general arithmetic series. For convenience, we will call the last term in the series b. Of course, we know that $b = a + (n - 1)d$.

Arithmetic Series (Cont.)

$$\begin{array}{ccccccccccccc} S & = & a & + & a+d & + & a+2d & + & \cdots & + & a+(n-1)d \\ S & = & b & + & b-d & + & b-2d & + & \cdots & + & b-(n-1)d \\ \hline 2S & = & a+b & + & a+b & + & a+b & + & \cdots & + & a+b \end{array}$$

Therefore: $\boxed{S = n\frac{a+b}{2}}$ In words:

> The sum of an arithmetic series is equal to the average of the first and last terms, multiplied by the number of terms.

This formula should work for the triangular numbers also, since they are arithmetic series. For T_n, we have the first term is 1, the last term is n, and the number of terms is n. Thus, according to the above formula we should have

$$T_n = n\frac{n+1}{2}$$

which agrees with our earlier result.