Solutions: AMC Prep for ACHS: Triangles

ACHS Math Competition Team

3 Feb 2009

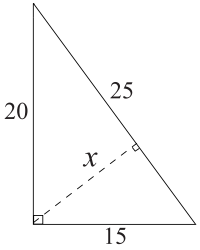
Problem 1

The sides of a triangle have lengths 15, 20, and 25. Find the length of the shortest altitude.

Problem 1

The sides of a triangle have lengths 15, 20, and 25. Find the length of the shortest altitude.

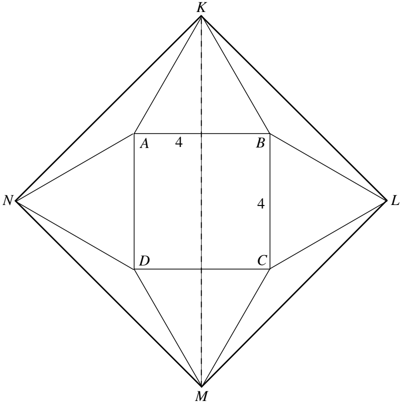
First notice that this is a right triangle, so two of the altitudes are the legs, whose lengths are 15 and 20. The third altitude, whose length is x, is the one drawn to the hypotenuse. The area of the triangle is $\frac{1}{2}(15)(20) = 150$. Using 25 as the base and x as the altitude, we have



$$\frac{1}{2}25x = 150 \implies x = \frac{300}{25} = 12$$

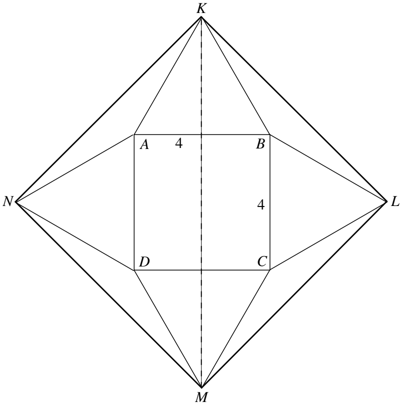
Problem 2

Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



Problem 2

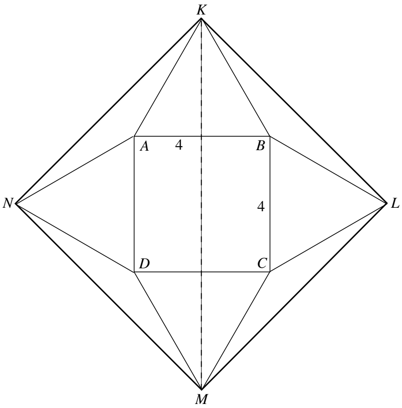
Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



Quadrilateral $KLMN$ is a square because it has 90° rotational symmetry. which implies that each pair of adjacent sides is congruent and perpendicular.

Problem 2

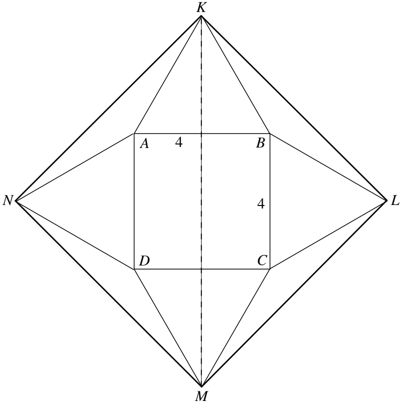
Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



Quadrilateral $KLMN$ is a square because it has 90° rotational symmetry. which implies that each pair of adjacent sides is congruent and perpendicular. Since $ABCD$ has sides of length 4 and K is $2\sqrt{3}$ from side $\overline{AB}$, the length of the diagonal $\overline{KM}$ is $4 + 4\sqrt{3}$.

Problem 2

Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



Quadrilateral $KLMN$ is a square because it has 90° rotational symmetry. which implies that each pair of adjacent sides is congruent and perpendicular. Since $ABCD$ has sides of length 4 and K is $2\sqrt{3}$ from side $\overline{AB}$, the length of the diagonal $\overline{KM}$ is $4 + 4\sqrt{3}$. Thus the area is

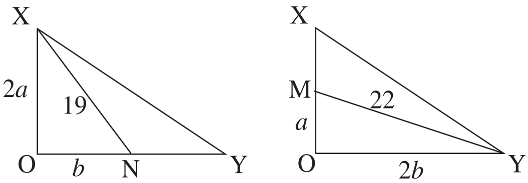
$$\frac{1}{2}\left(4 + 4\sqrt{3}\right)^2 = 32 + 16\sqrt{3}$$

Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

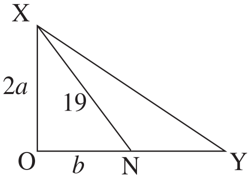
Problem 3

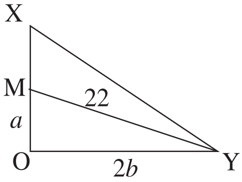
Let ΔXOY be a right triangle with $m\angle XOY = 90^{\circ}$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



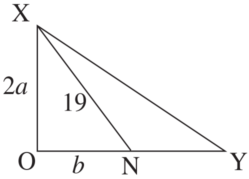


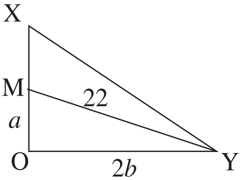
Let $OM = a$ and $ON = b$. Then

$19^2 = (2a)^2 + b^2$,

Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



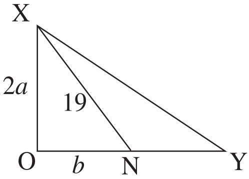


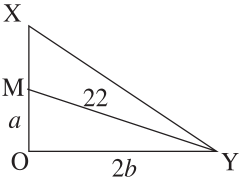
Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2, \quad 22^2 = a^2 + (2b)^2,$$

Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



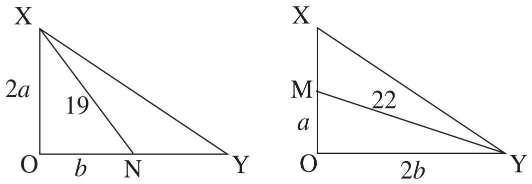


Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2, \quad 22^2 = a^2 + (2b)^2, \implies 5(a^2 + b^2) = 19^2 + 22^2 = 845$$

Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



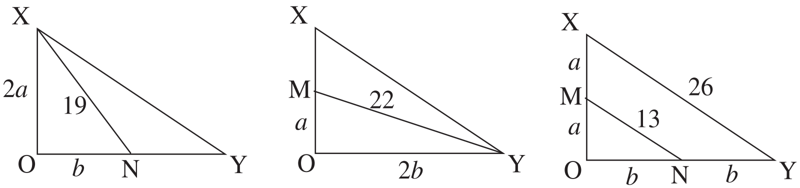
Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2, \quad 22^2 = a^2 + (2b)^2, \implies 5(a^2+b^2) = 19^2 + 22^2 = 845$$

$$MN = \sqrt{a^2 + b^2} = \sqrt{845/5} = \sqrt{169} = 13$$

Problem 3

Let ΔXOY be a right triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.



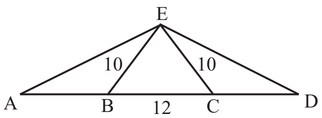
Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2, \quad 22^2 = a^2 + (2b)^2, \implies 5(a^2+b^2) = 19^2 + 22^2 = 845$$

$$MN = \sqrt{a^2 + b^2} = \sqrt{845/5} = \sqrt{169} = 13$$

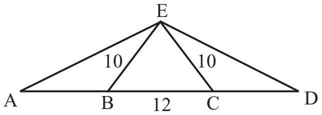
$$XY = 2MN = 26$$

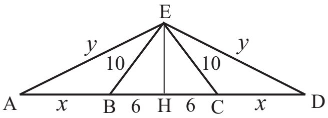
Problem 4



Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.

Problem 4

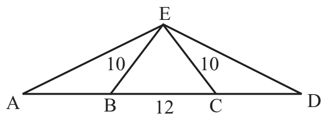




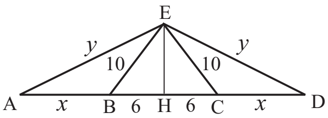
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.

Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Problem 4



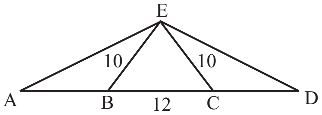
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



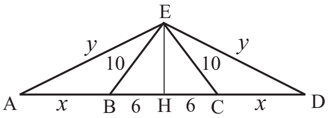
Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and

Problem 4



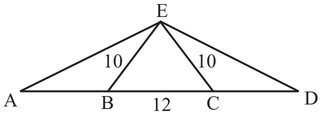
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



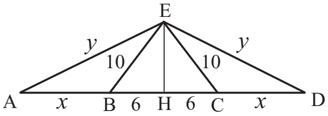
Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and $EH = \sqrt{10^2 - 6^2} = 8$.

Problem 4



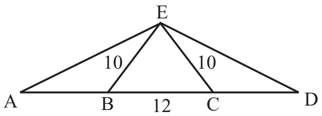
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



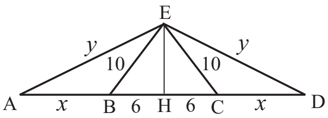
Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and $EH = \sqrt{10^2 - 6^2} = 8$. Let $AB = CD = x$ and $AE = ED = y$.

Problem 4



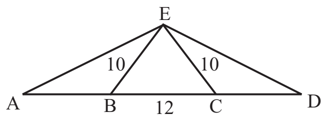
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



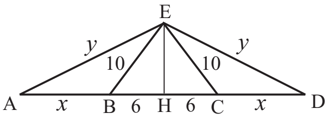
Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and $EH = \sqrt{10^2 - 6^2} = 8$. Let $AB = CD = x$ and $AE = ED = y$. Then $2x + 2y + 12 = 2(32)$,

Problem 4



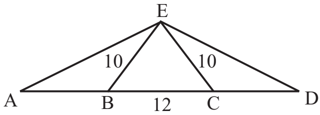
Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



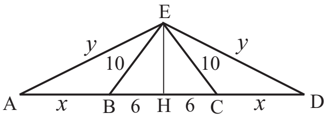
Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and $EH = \sqrt{10^2 - 6^2} = 8$. Let $AB = CD = x$ and $AE = ED = y$. Then $2x + 2y + 12 = 2(32)$, so $y = 26 - x$.

Problem 4



Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of ΔAED is twice that of ΔBEC. Find AB.

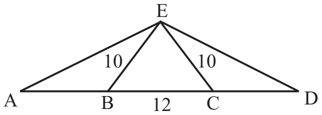


Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and ΔAED is isosceles.

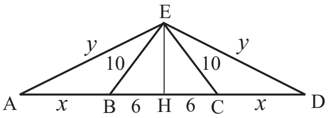
Segment $\overline{EH}$ is the common altitude of isosceles triangles ΔAED and ΔBEC, and $EH = \sqrt{10^2 - 6^2} = 8$. Let $AB = CD = x$ and $AE = ED = y$. Then $2x + 2y + 12 = 2(32)$, so $y = 26 - x$. Thus,

$$8^2 + (x + 6)^2 = y^2 = (26 - x)^2$$

Problem 4



Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice that of $\triangle BEC$. Find AB.



Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles.

Segment $\overline{EH}$ is the common altitude of isosceles triangles $\triangle AED$ and $\triangle BEC$, and $EH = \sqrt{10^2 - 6^2} = 8$. Let $AB = CD = x$ and $AE = ED = y$. Then $2x + 2y + 12 = 2(32)$, so $y = 26 - x$. Thus,

$$8^2 + (x+6)^2 = y^2 = (26 - x)^2$$

and $x = 9$.

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements.

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$,

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$, $\{4, 2, 1\}$,

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$, $\{4, 2, 1\}$, $\{3, 3, 1\}$,

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$, $\{4, 2, 1\}$, $\{3, 3, 1\}$, and $\{3, 2, 2\}$.

Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$, $\{4, 2, 1\}$, $\{3, 3, 1\}$, and $\{3, 2, 2\}$. Each of these must satisfy the triangle inequality: The length of any side must be less than the sum of the other two sides but greater than their difference. Of the above 4 combinations, 5 is not less than $1 + 1$,

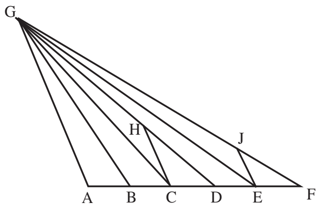
Problem 5

How many non-congruent triangles with perimeter 7 have integer side lengths?

First, compute the partitions of 7 (combinations of positive integers that add up to 7) containing three elements. These are: $\{5, 1, 1\}$, $\{4, 2, 1\}$, $\{3, 3, 1\}$, and $\{3, 2, 2\}$. Each of these must satisfy the triangle inequality: The length of any side must be less than the sum of the other two sides but greater than their difference. Of the above 4 combinations, 5 is not less than $1 + 1$, and 4 is not less than $2 + 1$. The remaining 2 combinations constitute valid side lengths.

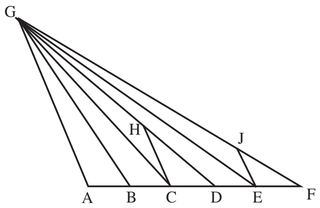
Problem 6

Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line $\overline{AF}$. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.



Problem 6

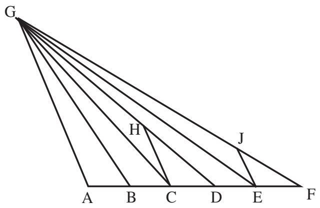
Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.



Since ΔAGD is similar to ΔCHD, we have $HC/1 = AG/3$.

Problem 6

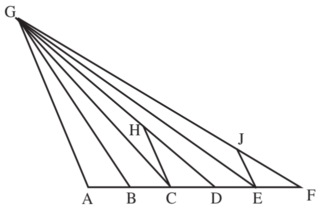
Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line $\overline{AF}$. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.



Since ΔAGD is similar to ΔCHD, we have $HC/1 = AG/3$. Also, ΔAGF is similar to ΔEJF, so $JE/1 = AG/5$.

Problem 6

Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line $\overline{AF}$. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.

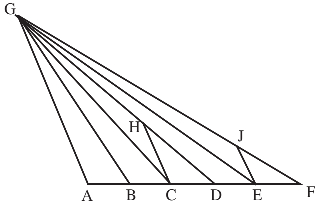


Since ΔAGD is similar to ΔCHD, we have $HC/1 = AG/3$. Also, ΔAGF is similar to ΔEJF, so $JE/1 = AG/5$. Therefore,

$$\frac{HC}{JE} = \frac{AG/3}{AG/5}$$

Problem 6

Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line $\overline{AF}$. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.



Since ΔAGD is similar to ΔCHD, we have $HC/1 = AG/3$. Also, ΔAGF is similar to ΔEJF, so $JE/1 = AG/5$. Therefore,

$$\frac{HC}{JE} = \frac{AG/3}{AG/5} = \frac{5}{3}.$$