Math League Contest #4, February 2, 1999

ACHS Math Competition Team

December 1, 2010

Problem 4-1

For what value of x does $\sqrt{45} = \sqrt{5} + \sqrt{x}$?

Problem 4-1

For what value of x does $\sqrt{45} = \sqrt{5} + \sqrt{x}$?

$$\sqrt{45} = 3\sqrt{5}$$

Problem 4-1

For what value of x does $\sqrt{45} = \sqrt{5} + \sqrt{x}$?

$$\sqrt{45} = 3\sqrt{5} = \sqrt{5} + 2\sqrt{5}$$

Problem 4-1

For what value of x does $\sqrt{45} = \sqrt{5} + \sqrt{x}$?

$$\sqrt{45} = 3\sqrt{5} = \sqrt{5} + 2\sqrt{5} = \sqrt{5} + \sqrt{20}$$

Problem 4-1

For what value of x does $\sqrt{45} = \sqrt{5} + \sqrt{x}$?

$$\sqrt{45} = 3\sqrt{5} = \sqrt{5} + 2\sqrt{5} = \sqrt{5} + \sqrt{20}$$

so $x = 20$

Problem 4-2

On day 1, a boy spent 30% of his $\$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with \$1 less than the amount he spent on day 1, what is the value of D?

Problem 4-2

On day 1, a boy spent 30% of his \$$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with \$1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$.

Problem 4-2

On day 1, a boy spent 30% of his \$$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with \$1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$. The remainder after day 2 is $(1 - 0.6) \times 0.7D = 0.4 \times 0.7D = 0.28D$.

Problem 4-2

On day 1, a boy spent 30% of his $$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with $1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$. The remainder after day 2 is $(1 - 0.6) \times 0.7D = 0.4 \times 0.7D = 0.28D$.

So

$$0.28D = 0.3D - 1$$

Problem 4-2

On day 1, a boy spent 30% of his $D weekly pay. On day 2, he spent 60% of the remainder. If this left him with $1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$. The remainder after day 2 is $(1 - 0.6) \times 0.7D = 0.4 \times 0.7D = 0.28D$.

So

$$0.28D = 0.3D - 1 \iff 1 = (0.3 - 0.28)D$$

Problem 4-2

On day 1, a boy spent 30% of his \$$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with \$1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$. The remainder after day 2 is $(1 - 0.6) \times 0.7D = 0.4 \times 0.7D = 0.28D$.

So

$$0.28D = 0.3D - 1 \iff 1 = (0.3 - 0.28)D = 0.02D$$

Problem 4-2

On day 1, a boy spent 30% of his $$D$ weekly pay. On day 2, he spent 60% of the remainder. If this left him with $1 less than the amount he spent on day 1, what is the value of D?

The remainder after day 1 is $(1 - 0.3)D = 0.7D$. The remainder after day 2 is $(1 - 0.6) \times 0.7D = 0.4 \times 0.7D = 0.28D$.

So

$$0.28D = 0.3D - 1 \iff 1 = (0.3 - 0.28)D = 0.02D \iff D = 50$$

Problem 4-3

Write, in simplest form, the value of the quotient

$$\frac{19^{98} + (342)(19^{97})}{19^{99}}.$$

Problem 4-3

Write, in simplest form, the value of the quotient

$$\frac{19^{98} + (342)(19^{97})}{19^{99}}.$$

Since $342 = 18 \times 19$, then

$$\frac{19^{98} + (342)(19^{97})}{19^{99}} = \frac{19^{98} + (18)(19^{98})}{19^{99}}$$

Problem 4-3

Write, in simplest form, the value of the quotient

$$\frac{19^{98} + (342)(19^{97})}{19^{99}}.$$

Since $342 = 18 \times 19$, then

$$\frac{19^{98} + (342)(19^{97})}{19^{99}} = \frac{19^{98} + (18)(19^{98})}{19^{99}} = \frac{1 + 18}{19}$$

Problem 4-3

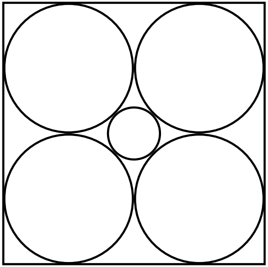
Write, in simplest form, the value of the quotient
$$\frac{19^{98} + (342)(19^{97})}{19^{99}}.$$

Since $342 = 18 \times 19$, then

$$\frac{19^{98} + (342)(19^{97})}{19^{99}} = \frac{19^{98} + (18)(19^{98})}{19^{99}} = \frac{1+18}{19} = 1$$

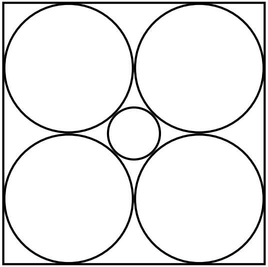
Problem 4-4

Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?



Problem 4-4

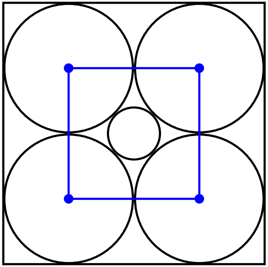
Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?



The large circles have diameter 2.

Problem 4-4

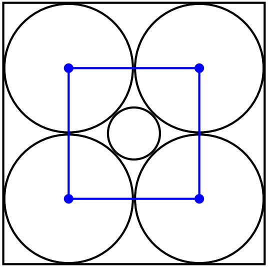
Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?



The large circles have diameter 2. Draw a square (of side length 2) joining the centers of the large circles.

Problem 4-4

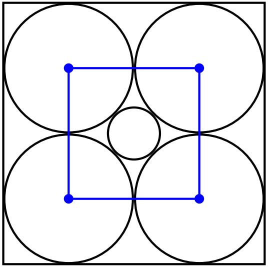
Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?



The large circles have diameter 2. Draw a square (of side length 2) joining the centers of the large circles. The diagonal of the square has length $2\sqrt{2}$, so if the radius of the small circle is r, then

Problem 4-4

Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?

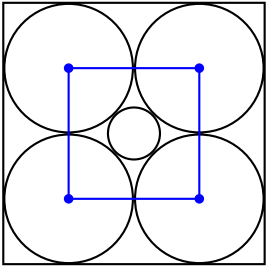


The large circles have diameter 2. Draw a square (of side length 2) joining the centers of the large circles. The diagonal of the square has length $2\sqrt{2}$, so if the radius of the small circle is r, then

$$2\sqrt{2} = 2 + 2r$$

Problem 4-4

Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two of the other circles. A smaller fifth circle is drawn tangent to each of the other four circles, as shown. How long is the radius of the fifth circle?



The large circles have diameter 2. Draw a square (of side length 2) joining the centers of the large circles. The diagonal of the square has length $2\sqrt{2}$, so if the radius of the small circle is r, then

$$2\sqrt{2} = 2 + 2r \implies r = \sqrt{2} - 1$$

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2 y^2 = a$?

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2 y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum.

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Method II

$y^2 = 1 - x^2$ so $a = x^2y^2 = x^2(1 - x^2) = x^2 - x^4$.

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Method II

$y^2 = 1 - x^2$ so $a = x^2y^2 = x^2(1 - x^2) = x^2 - x^4$. So $x^4 - x^2 = -a$. Completing the square, this can be written

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Method II

$y^2 = 1 - x^2$ so $a = x^2y^2 = x^2(1 - x^2) = x^2 - x^4$. So $x^4 - x^2 = -a$. Completing the square, this can be written

$$\left(x^2 - \frac{1}{2}\right)^2 - \frac{1}{4} = -a$$

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Method II

$y^2 = 1 - x^2$ so $a = x^2y^2 = x^2(1 - x^2) = x^2 - x^4$. So $x^4 - x^2 = -a$. Completing the square, this can be written

$$\left(x^2 - \frac{1}{2}\right)^2 - \frac{1}{4} = -a \iff a = \frac{1}{4} - \left(x^2 - \frac{1}{2}\right)^2$$

Problem 4-5

What is the greatest possible value of a for which there is at least one real solution (x, y) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

Method I

If two numbers have a fixed sum, their product is maximized when each is equal to half that sum. Since $x^2 + y^2 = 1$, when $x^2 = y^2 = \frac{1}{2}$, the product $a = x^2y^2$ takes its maximum value, $\frac{1}{4}$.

Method II

$y^2 = 1 - x^2$ so $a = x^2y^2 = x^2(1 - x^2) = x^2 - x^4$. So $x^4 - x^2 = -a$. Completing the square, this can be written

$$\left(x^2 - \frac{1}{2}\right)^2 - \frac{1}{4} = -a \iff a = \frac{1}{4} - \left(x^2 - \frac{1}{2}\right)^2$$

The maximum possible value for a is $\frac{1}{4}$, which occurs when $x^2 = 1/2$.

Problem 4-6

A hound was chasing a fox. Whenever the hound took 4 leaps, the fox took 5. If 3 hound leaps cover the same distance as 4 fox leaps, and the fox's head start was equal to 90 hound leaps, how many leaps did the hound need to catch the fox?

Problem 4-6

A hound was chasing a fox. Whenever the hound took 4 leaps, the fox took 5. If 3 hound leaps cover the same distance as 4 fox leaps, and the fox's head start was equal to 90 hound leaps, how many leaps did the hound need to catch the fox?

Let the distance of a hound leap be h and that of a fox leap be f. Then $3h = 4f$, or $f = 3h/4$.

Problem 4-6

A hound was chasing a fox. Whenever the hound took 4 leaps, the fox took 5. If 3 hound leaps cover the same distance as 4 fox leaps, and the fox's head start was equal to 90 hound leaps, how many leaps did the hound need to catch the fox?

Let the distance of a hound leap be h and that of a fox leap be f. Then $3h = 4f$, or $f = 3h/4$. The head start distance is $90h$.

Problem 4-6

A hound was chasing a fox. Whenever the hound took 4 leaps, the fox took 5. If 3 hound leaps cover the same distance as 4 fox leaps, and the fox's head start was equal to 90 hound leaps, how many leaps did the hound need to catch the fox?

Let the distance of a hound leap be h and that of a fox leap be f. Then $3h = 4f$, or $f = 3h/4$. The head start distance is $90h$. When the hound takes 4 leaps to cover a distance of $4h$, the fox takes 5 leaps to cover a distance of $5f = 15h/4$. So with each leap, the hound reduces the gap by $h - 15h/16 = h/16$.

Problem 4-6

A hound was chasing a fox. Whenever the hound took 4 leaps, the fox took 5. If 3 hound leaps cover the same distance as 4 fox leaps, and the fox's head start was equal to 90 hound leaps, how many leaps did the hound need to catch the fox?

Let the distance of a hound leap be h and that of a fox leap be f. Then $3h = 4f$, or $f = 3h/4$. The head start distance is $90h$. When the hound takes 4 leaps to cover a distance of $4h$, the fox takes 5 leaps to cover a distance of $5f = 15h/4$. So with each leap, the hound reduces the gap by $h - 15h/16 = h/16$. So the number of leaps needed to catch the fox is $16 \times 90 = 1440$.