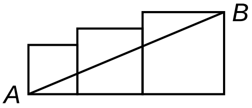
Math League Contest #3, January 5, 1999

ACHS Math Competition Team

November 10, 2010

Problem 3-1

Three squares are lined up horiontally, as shown. The area of the first, second, and third squares is 9, 16, and 25, resp. How long is $\overline{AB}$?



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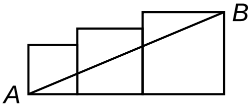
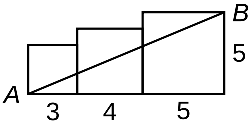


$$AB = \sqrt{5^2 + 12^2} = 13$$



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If a, b, and c are three different positive integers that satisfy

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what is the least possible value of the sum $a + b + c$?

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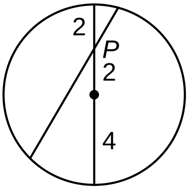
Note that

$$(\sqrt{a})^4 = (a^{1/2})^4 = a^{4/2} = a^2$$

and similarly for the other two terms. So we seek the smallest sum where $a^2 + b^2 = c^2$ which occurs for the integers $(a, b, c) = (3, 4, 5)$ whose sum is $3 + 4 + 5 = 12$.

Problem 3-3

Point P splits a diameter of a circle into segments of length 2 and 6. What is the shortest distance from the center of the circle to a chord through P that makes a 30° angle with the diameter, as shown?



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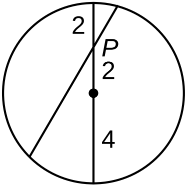
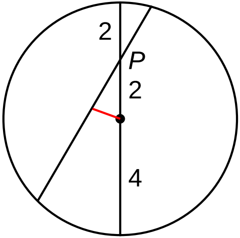


Draw a perpendicular from the circle's center to the chord, creating a 30-60-90 triangle, with hypotenuse of length 2. The short leg is half the hypotenuse, or 1.



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from which we conclude that $3n - 1998 = 0$ or $n = 666$.

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$$x - 4 = \pm 1, \pm 2, \pm 4 \implies x = 0, 2, 3, 5, 6, \text{ or } 8.$$

Only $x = 0$ and $x = 3$ satisfy both requirements, and both also make $x/(x-6)$ integral.

Problem 3-6

Rowing upstream, Pat dropped a hat into the river. Ten minutes later, he reversed direction, rowed downstream, and retrieved the hat 1 km downstream from where he had dropped it. If the river flows at a constant rate, and Pat rows at a constant rate (relative to the river), what is the river's rate, in km/hr?

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Consider that Pat is walking along a long conveyor belt, when he drops the hat and walks for ten minutes. When he turns around, it will take another ten minutes to return to the hat. The conveyor belt moves 1 km in 20 minutes, or in $\frac{1}{3}$ hr, so its velocity is 3 km/hr.