Math League Contest #2, December 1, 1998

ACHS Math Competition Team

October 13, 2009

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The next possibility is $(1, 2, 2)$ which does represent the side lengths of a triangle, with perimeter 5.

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$$1 \times 2 \times 37 \times 27 = \boxed{1998}$$

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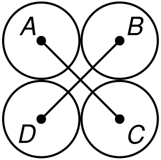
From the quadratic formula, the roots are

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The positive root is $\boxed{\dfrac{1+\sqrt{5}}{2}}$.

Problem 2-4

A, B, C, and D are the centers of four congruent circles, tangent to each other as shown. If $AC = BD = 12$, what is the area of one of the four circles?



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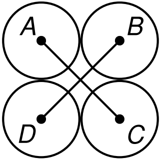

Each of the two red dashed lines has length $2r$, where r is the radius of the circles. Applying the Pythagorean theorem to $\triangle ABC$, we obtain

$$(2r)^2+(2r)^2 = 12^2 \implies 8r^2 = 144 \implies r^2 = 18.$$

Therefore, the area of each of the circles is

$$\pi r^2 = \boxed{18\pi}$$

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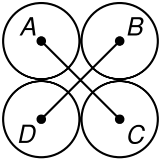

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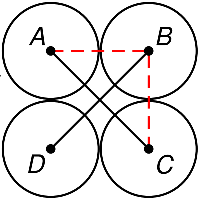

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Or simply note that $\sqrt{2} \times 2r = 12$ and proceed to the same solution.

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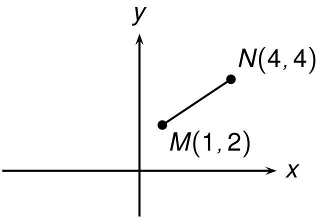
The set of x satisfying both these conditions can be written as

$$\boxed{\left\{x \mid -\frac{7}{9} < x \le 1\right\} \cup \{x \mid x \ge 2\}}$$

Problem 2-6

The area of a triangle is 5. Two of its vertices are at $M(1, 2)$ and $N(4, 4)$. Its third vertex A is on the x-axis. What are all possible coordinates of this 3rd vertex?



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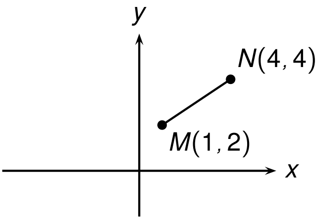
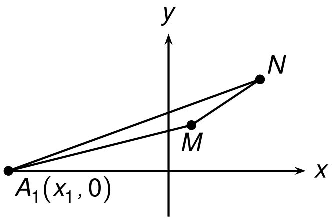


How to compute area of triangle MNA_1?



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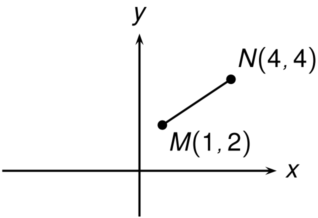
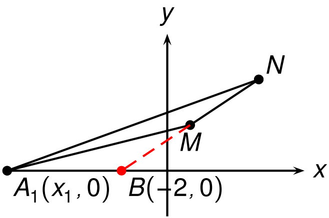


How to compute area of triangle MNA_1? Extend $\overline{NM}$ till it meets the x-axis at $B:(-2,0)$. Then

$$5 = \text{Area}(\Delta MNA_1)$$



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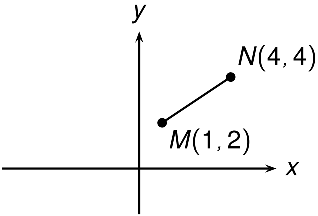

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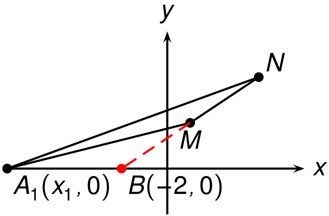

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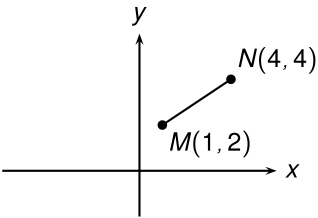

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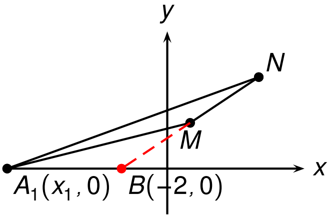

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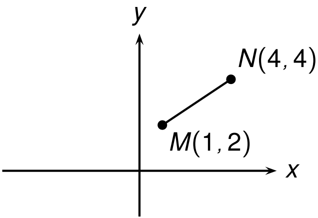

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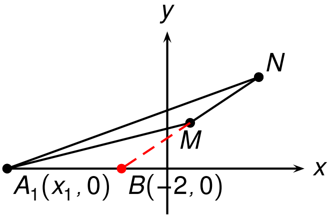

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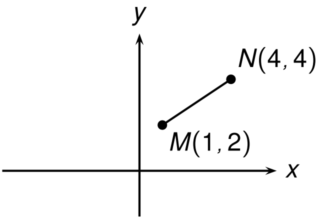

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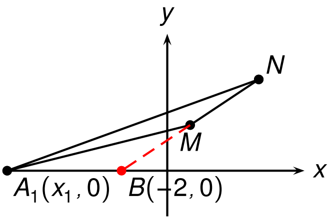

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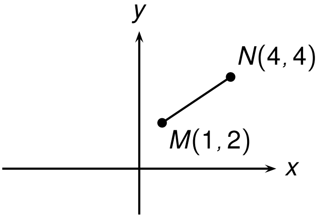

Similarly, if A_2 is placed to the right of B, then

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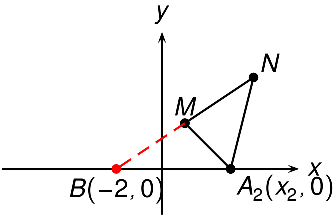

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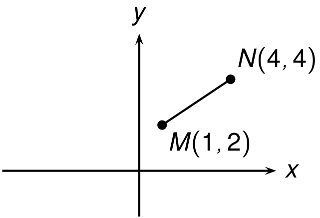

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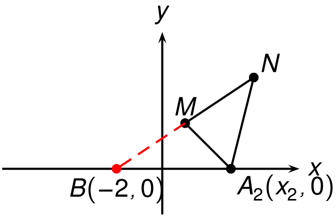

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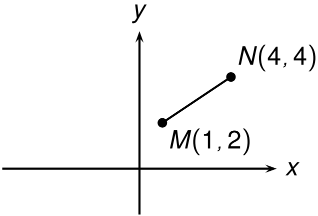

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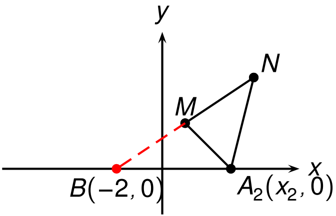

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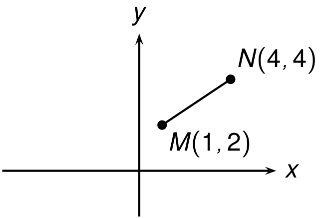

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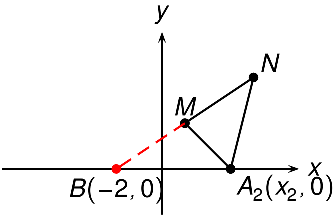

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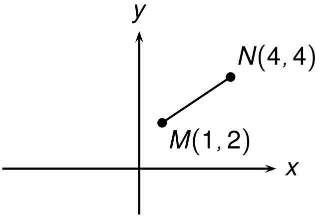

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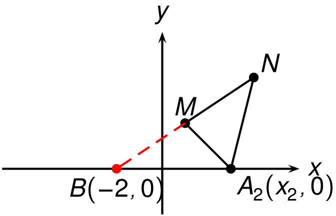

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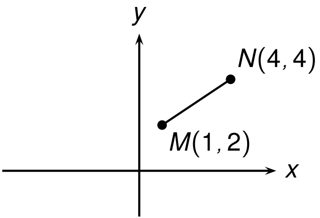

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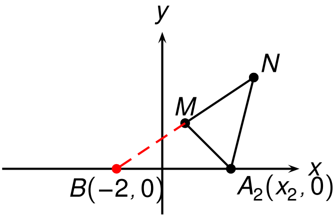

so that $x_2 = 3$. The solution is therefore the pair of points $(-7,0)$ and $(3,0)$.