Math League Contest #1, October 27, 1998

ACHS Math Competition Team

November 10, 2010

Problem 1-1

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If 3 of the talks are 1 hour each, then the 4th talk will be as long as possible: 2 hours or 120 minutes.

Problem 1-2

Whenever the number $66\ldots66$, which consists of only 6's, has twice as many digits as the number $33\ldots33$, which consists of only 3's, the product $(66\ldots66)(33\ldots33)$ does **not** contain the digit d. What are all five possible values of d.

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Since $6666 \times 33 = 219978$ we see that the product can contain the digits 1, 2, 7, 8, and 9. Since we are told that there are five digits which can not appear, they must be the remaining five digits: 0, 3, 4, 5, and 6.

Problem 1-3

The line $2y - 3x = 12$ intersects the x-axis at A and the y-axis at B. For what value of $k > 0$ will a line through B intersect the x-axis at $C : (k, 0)$ so that the area of $\triangle ABC$ is 21?

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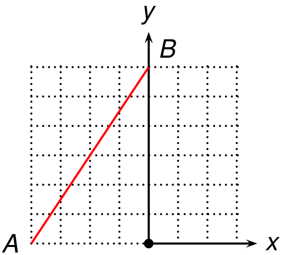

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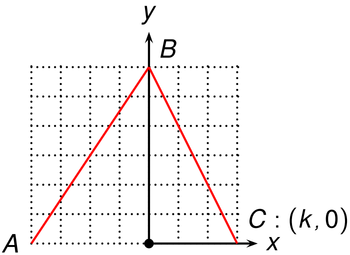
$$\text{Area } \Delta ABC = 21 = \frac{1}{2} \times 6 \times (4 + k)$$



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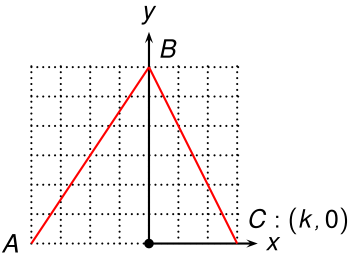
$$k = \frac{2 \times 21}{6} - 4 = 3$$



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so that

$$m - h = 7$$

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Solving the original equation for x^3 yields $x^3 = -78x - 666$. This is true for each of the roots, so

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$$a^3 + b^3 + c^3 = -78(a + b + c) - 3(666) = -3(666) = -1998$$

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For a non-leap year, there are 7 groupings: {Jan, Oct}, {Feb, Mar, Nov}, {Apr, Jul}, {May}, {Jun}, {Aug}, and {Sep, Dec}.

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For both leap and non-leap years, the groups contain at least 1 month and at most 3 months. Therefore $(m, M) = (1, 3)$