ACHS Math Team
Lecture: Introduction to Set Theory

Peter S. Simon

Introduction to Set Theory

A *set* is a collection of objects, called *elements* or *members* of the set. We will usually denote a set by a capital letter such as A, B, or C, and an element of a set by a lower-case letter such as a, b, c. Sets are usually denoted by listing their contents between curly braces, as in $S = \{a, b, c\}$. If x is an element of the set A, then we write $x \in A$. A set can be defined either by explicitly listing its elements (roster method) or by giving a rule for membership (the property method).

Example (Roster and Property Methods)

The set S of all decimal digits can be defined by the roster method as

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$$

or by the property method as in

$$S = \{x \,|\, x \text{ is a decimal digit}\}.$$

We read this as "S is the set of all x such that x is a decimal digit."

Some Examples of Sets

- The set of days of the week: $\{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$ is an example of a finite set.
- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$ is an example of an infinite set.
- The set of integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is an example of an infinite set.
- The set of dogs with sixty legs is an example of an empty set.

Set Membership

The order in which the elements of a set is listed does not matter. All of the following define the same set:

$$A = \{a, b, c, d\}, \quad A = \{b, d, c, a\}, \quad A = \{d, c, b, a\}$$

It is true that

$$a \in A, \quad b \in A, \quad e \notin A$$

Remember: $\in$ means “is a member of” and $\notin$ means “is not a member of”

Subsets

If every element of a set A is also an element of the set B, we say that A is a *subset* of B and write $A \subset B$. Note that for any set A it is true that $A \subset A$.
If $A \subset B$ and $B \subset A$, then the sets contain exactly the same elements and we say that A and B are *equal*, written as $A = B$. Otherwise, we may write $A \neq B$.
If $A \subset B$ but $A \neq B$ then we say that A is a *proper* subset of B.

Example

$$A = \{a, b, c, d\}, \quad B = \{a, b, c\}, \quad C = \{c, d, a, b\}$$

Since every element of B is also an element of A, then $B \subset A$. In fact, B is a proper subset of A since A is not a subset of B.
Since $A \subset C$ and $C \subset A$ then $A = C$.

Universal Set and Empty Set

We often find it convenient to restrict our attention to subsets of some particular set which we refer to as the *universe*, *universal set*, or *universal space* denoted by $\mathcal{U}$. Elements of $\mathcal{U}$ are often referred to as *points* of the space.

We call the set containing no elements the *empty set* or the *null set* and denote it by $\emptyset = \{\}$. It is a subset of every set.

Venn Diagrams

A universe $\mathcal{U}$ can be represented geometrically by the set of points inside a rectangle. We represent subsets of $\mathcal{U}$ as sets of points inside circles. Such diagrams are called *Venn diagrams* and are useful in visualizing relationships between sets.

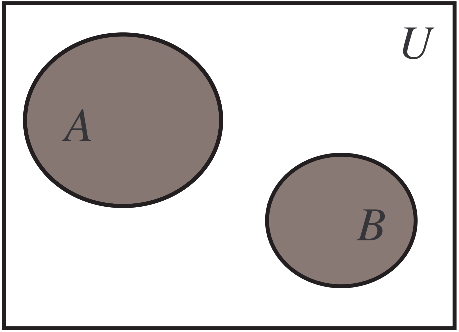
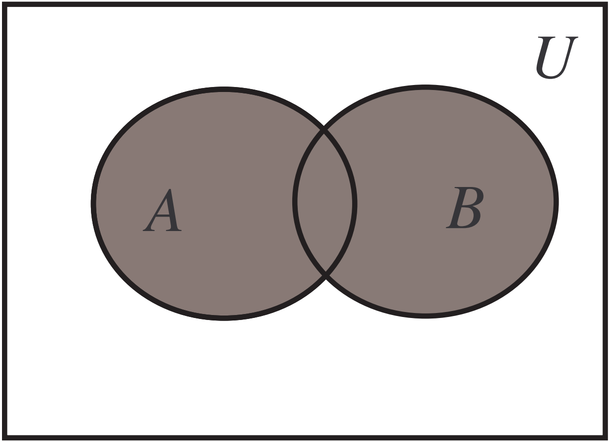


Figure: Venn diagram.

Union

The set of all points belonging to either set A or set B or to both sets A and B is called the *union* of A and B and is denoted by $A \cup B$ (shaded below).



Set Union Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

$$A \cup B =$$

Set Union Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

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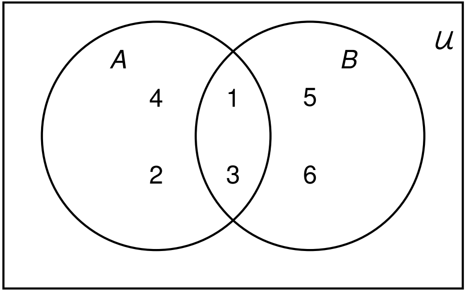
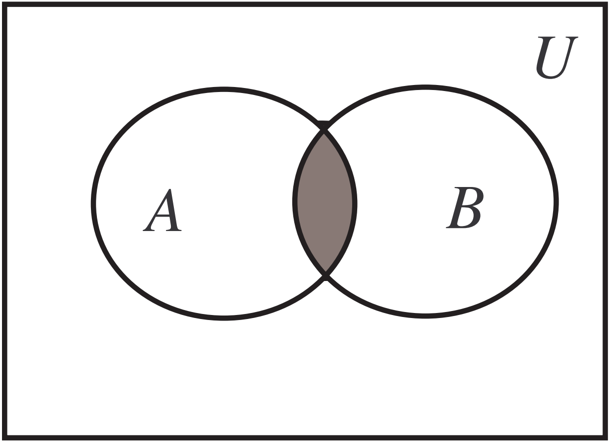
We can represent these sets using a Venn diagram:



Intersection

The set of all points belonging simultaneously to both sets A and B is called the *intersection* of A and B and is denoted by $A \cap B$ (shaded below).



Set Intersection Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

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Set Intersection Example

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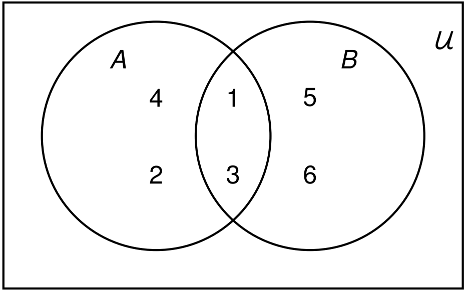
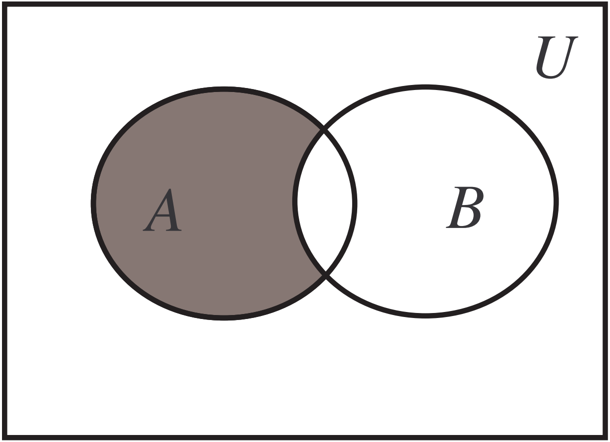


Difference

The set consisting of all elements of A that do not belong to B is called the *difference* of A and B and is denoted by $A - B$ (shaded below).



Set Difference Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

$$A - B =$$

Set Difference Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

$$A - B = \{2, 4\},$$

$$B - A =$$

Set Difference Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}$$

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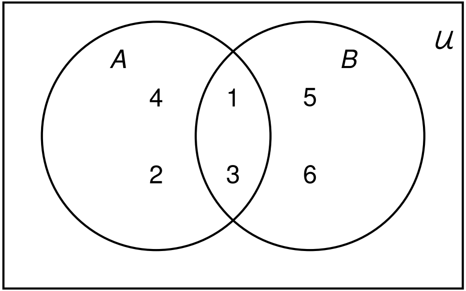
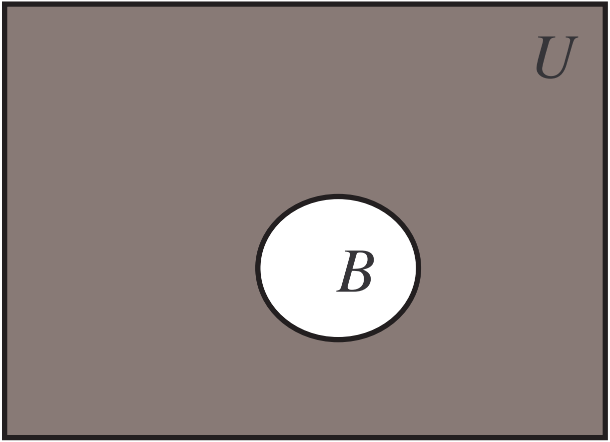
$$B - A = \{5, 6\}$$

We can represent these sets using a Venn diagram:



Complement

The set consisting of all elements of U that do not belong to B is called the *complement* of B and is denoted by B' (shaded below).



Set Complement Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}, \quad \mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$A' =$$

Set Complement Example

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}, \quad \mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$A' = \{5, 6, 7, 8\},$$

$$B' =$$

Set Complement Example

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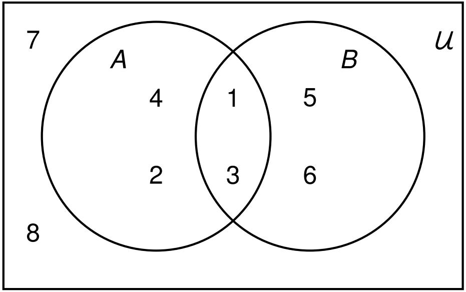
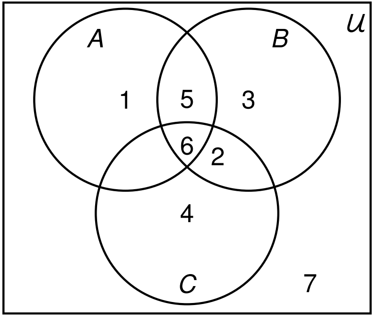
$$B' = \{2, 4, 7, 8\}$$

We can represent these sets using a Venn diagram:



Venn Diagram for Three Sets A, B, and C

$$A = \{1, 5, 6\}, \quad B = \{2, 3, 5, 6\}, \quad C = \{2, 4, 6\}$$



$A' =$

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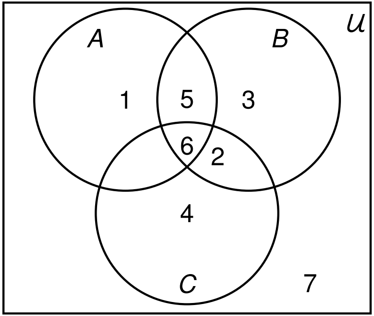

$A' = \{2, 3, 4, 7\}$
$A \cap B =$

Venn Diagram for Three Sets A, B, and C

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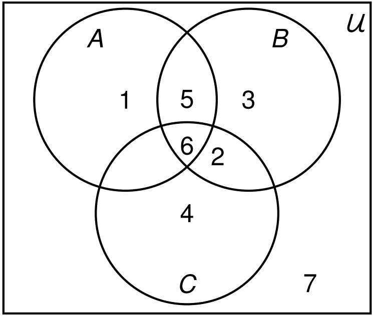

$A' = \{2, 3, 4, 7\}$
$A \cap B = \{6, 5\}$
$A \cap B \cap C =$

Venn Diagram for Three Sets A, B, and C

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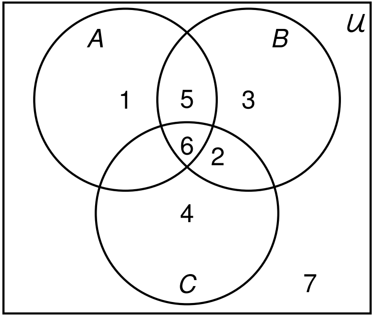

$A' = \{2, 3, 4, 7\}$
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Venn Diagram for Three Sets A, B, and C

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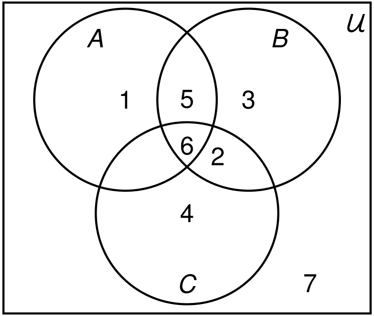

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$(A \cup B) - C =$

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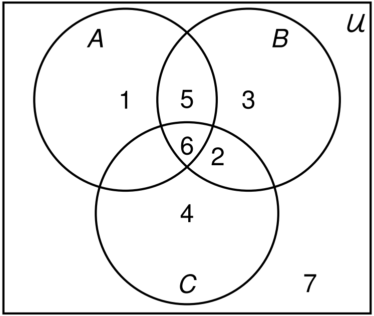

$A' = \{2, 3, 4, 7\}$
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$A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$
$(A \cup B) - C = \{1, 5, 3\}$
$C \cup C' =$

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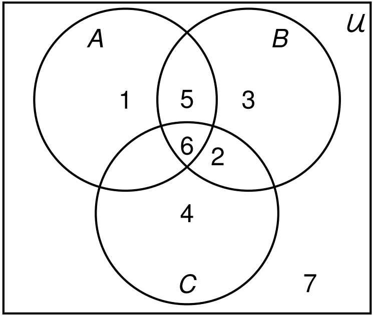

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$A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$
$(A \cup B) - C = \{1, 5, 3\}$
$C \cup C' = \{1, 2, 3, 4, 5, 6, 7\} = \mathcal{U}$
$C \cap C' =$

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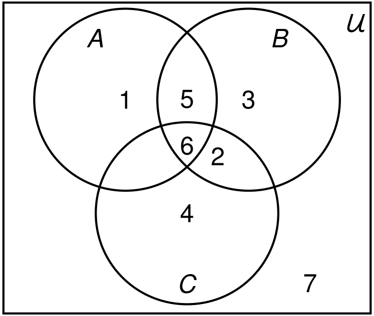

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$(A \cup B) - C = \{1, 5, 3\}$
$C \cup C' = \{1, 2, 3, 4, 5, 6, 7\} = \mathcal{U}$
$C \cap C' = \{\} = \emptyset$

Word Problems Into Set Notation

Consider the universe consisting of the students Alice, Bob, Charles, Dick, Emily, and Frank. Define the following sets:

- M The set of male students.
- F The set of female students.
- O The set of students 13 and older.
- P The set of students having PE for first period.

Translate the following events into mathematical set notation:

1. Male students having PE in the first period.
2. Students under 13 years of age who do not have PE first period.
3. The set of students who are female or who are older than 12 (or both).
4. The set of students who do not have PE first period, or are 12 or younger.

Set Theory Problems

Let the universe U be the members of the United Federation of Planets' Star Fleet. Assume that members are either terrestrial (earth-men) or Vulcan. Let V be the set of Vulcan members of Star Fleet. Let A be the set of Star Fleet members who graduated from the Academy. Let O be the set of StarFleet Officers. Let K be the set of StarFleet members whose last names are "Kirk".

1. Translate the following descriptions into mathematical set notation:
   - 1.1 Vulcans whose last names are "Kirk."
   - 1.2 Officers who graduated from the Academy.
   - 1.3 Enlisted members (nonofficers) whose last name is not "Kirk."
   - 1.4 Humans who did not graduate from the academy but became officers anyway.
   - 1.5 Members who are either officers or Vulcans, but graduated from the academy.

More Set Theory Problems (Cont.)

V: Vulcan members of Star Fleet.
A: Academy graduates.
O: Officers.
K: Last names are "Kirk"
Translate the following mathematical set descriptions into word descriptions:

1. $O - V$

2. $O \cap V'$

3. $V \cup O'$

4. $V \cup V'$

The Number of Elements in a Set

Suppose S is a set. Then let us define $n(S)$ to be the number of elements in the set S.

Example: Let $A = \{a, b, c, d\}$ and $B = \{c, d, e, f, g\}$. Then $n(A) = 4$ and $n(B) = 5$.

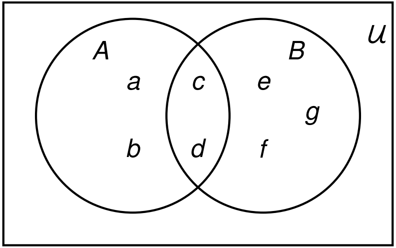
Now suppose we want to know $n(A \cup B)$. Since $A \cup B = \{a, b, c, d, e, f, g\}$, then $n(A \cup B) = 7$. Note that

$$n(A \cup B) = 7 \neq n(A) + n(B) = 4 + 5 = 9.$$

Why not? (Draw Venn diagram).

The Number of Elements in a Union of 2 Sets

Let's draw a Venn diagram for the previous example.



If we try to count the number of elements in $A \cup B$ as $n(A) + n(B) = 4 + 5 = 9$, we see that we have overcounted because we have counted the elements in the intersection $A \cap B$ twice. Therefore, we have to subtract this and the final formula is

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

Number of Elements in a Set Union: Example

All the students at ACHS must take at least one elective, choosing from art or drama. There are a total of 35 students, with 15 students enrolled in drama and 26 students enrolled in art. How many students are simultaneously enrolled in both art and science?

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Solution

Let A be the set of students enrolled in art, and D the set of students enrolled in drama.

$$n(A \cup D) = n(A) + n(D) - n(A \cap D)$$

We are told that $n(A \cup D) = 35$, $n(A) = 26$, $n(D) = 15$. So the above equation becomes

$$35 = 26 + 15 - n(A \cap D) \implies n(A \cap D) = 26 + 15 - 35 = 6$$

Problem 5-2 (2002) Revisited

Of 50 students, 20 take art and 25 take music. If 10 students take both subjects, how many take neither?

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We have $n(U) = 50$, $n(A) = 20$, $n(M) = 25$, $n(A \cap M) = 10$.

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$$n((A \cup M)') = n(U) - n(A \cup M)$$

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$$\begin{aligned} n((A \cup M)') &= n(U) - n(A \cup M) \\ &= n(U) - [n(A) + n(M) - n(A \cap M)] \\ &= 50 - (20 + 25 - 10) = 15 \end{aligned}$$

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$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{a, b\}$$

(Recall that the empty set is a subset of any set, and that any set is a subset of itself.) When constructing a subset of S, we can choose to include a or not (2 choices) and we can choose to include b or not (2 choices). So there are $2 \times 2 = 4$ subsets of S, or of any set containing 2 elements.

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$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{a, b\}, \quad \{a, c\}, \quad \{b, c\}, \quad \{a, b, c\}$$

Note that there are $2 \times 2 \times 2 = 2^3 = 8$ subsets of the set T, and in fact there are 8 subsets of any set containing 3 elements.

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The number of subsets of any set containing $\boldsymbol{n}$ elements is $\mathbf{2^n}$.

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$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{a, b\}, \quad \{a, c\}, \quad \{b, c\}, \quad \{a, b, c\}$$

Note that there are $2 \times 2 \times 2 = 2^3 = 8$ subsets of the set T, and in fact there are 8 subsets of any set containing 3 elements.
The number of subsets of any set containing $\boldsymbol{n}$ elements is $\mathbf{2^n}$. Another way: add up the number of subsets with $0, 1, 2, \ldots, n$ elements:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

Some Set Theorems

$A \cup B = B \cup A$	Commutative law for unions	(1)
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law for unions	(2)
$A \cap B = B \cap A$	Commutative law for intersections	(3)
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative law for intersections	(4)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	First distributive law	(5)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Second distributive law	(6)
$A - B = A \cap B'$		(7)
$A \subset B \implies B' \subset A'$		(8)
$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset, \quad A \cup \mathcal{U} = \mathcal{U}, \quad A \cap \mathcal{U} = A$		(9)
$(A \cup B)' = A' \cap B'$	De Morgan's first law	(10)
$(A \cap B)' = A' \cup B'$	De Morgan's second law	(11)
$A = (A \cap B) \cup (A \cap B')$	for any sets A and B	(12)

The Number of Elements in a Union of 3 Sets

$n(A \cup B \cup C)$

The Number of Elements in a Union of 3 Sets

$$
\begin{aligned}
& n(A \cup B \cup C) \\
& \qquad = n([A \cup B] \cup C)
\end{aligned}
$$

The Number of Elements in a Union of 3 Sets

$$\begin{aligned} &n(A \cup B \cup C) \\ &\qquad = n([A \cup B] \cup C) \\ &\qquad = n(A \cup B) + n(C) - n([A \cup B] \cap C) \end{aligned}$$

The Number of Elements in a Union of 3 Sets

$$
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&\quad = n([A \cup B] \cup C) \\
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&\quad = n(A \cup B) + n(C) - n([A \cap C] \cup [B \cap C])
\end{aligned}
$$

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& \quad = n(A \cup B) + n(C) - n([A \cup B] \cap C) \\
& \quad = n(A \cup B) + n(C) - n([A \cap C] \cup [B \cap C]) \\
& \quad = n(A) + n(B) - n(A \cap B) + n(C) - [n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)]
\end{aligned}
$$

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$$
\begin{aligned}
&n(A \cup B \cup C) \\
&\quad = n([A \cup B] \cup C) \\
&\quad = n(A \cup B) + n(C) - n([A \cup B] \cap C) \\
&\quad = n(A \cup B) + n(C) - n([A \cap C] \cup [B \cap C]) \\
&\quad = n(A) + n(B) - n(A \cap B) + n(C) - [n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)] \\
&\quad = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)
\end{aligned}
$$