Solutions: AMC Prep for ACHS: Counting and Probability

ACHS Math Competition Team

5 Jan 2009

Problem 1

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The factors of 60 are

$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, \text{ and } 60.$$

Six of the twelve factors are less than 7, so the probability is $1/2$.

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Of the $\frac{100}{2} = 50$ integers that are divisible by 2, there are $\left\lfloor \frac{100}{6} \right\rfloor = \left\lfloor 16\frac{2}{3} \right\rfloor = 16$ that are divisible by both 2 and 3. So there are $50 - 16 = 34$ that are divisible by 2 and not by 3, and $34/100 = 17/50$.

Problem 3

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$$\left(\frac{1}{10}\right)\left(1+\frac{9}{10}+\frac{8}{10}+\frac{6}{10}+\frac{4}{10}+\frac{2}{10}+\frac{1}{10}+0+0+0\right)$$

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$$\left(\frac{1}{10}\right)\left(1 + \frac{9}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} + 0 + 0 + 0\right) = \frac{2}{5}$$

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The last "word," which occupies position 120, is USOMA. Immediately preceding this we have USOAM, USMOA, USMAO, USAOM,and USAMO. The alphabetic position of the word USAMO is consequently 115.

Problem 5

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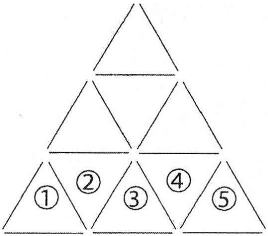
A point P is chosen randomly in the interior of an equilateral triangle ABC. What is the probability that ΔABP has a greater area than each of ΔACP and ΔBCP?

By symmetry, each of ΔABP, ΔACP, and ΔBCP is largest with the same probability. Since the sum of these probabilities is 1, the probability must be $1/3$ for each.

Problem 6

A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks are needed to construct a large equilateral triangle if the base row consists of 2003 small equilateral triangles?



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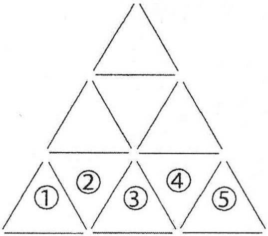

The base row will contain 1002 upward-pointing triangles and 1002 downward-pointing triangles. The number N of toothpicks is 3 times the number of upward-pointing small triangles:

$$N = 3(1 + 2 + 3 + \cdots + 1002)$$

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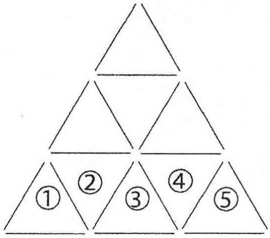

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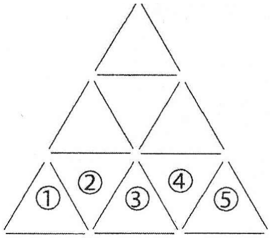

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$$N = 3(1 + 2 + 3 + \cdots + 1002) = 3\frac{(1002)(1003)}{2} = 1,507,509$$

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Juan rolls a fair, regular, octahedral die marked with the numbers 1 through 8. Then Amal rolls a fair six-sided die. What is the probability that the product of the two rolls is a multiple of 3?

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$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

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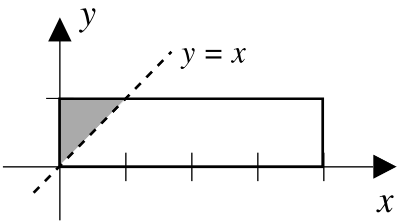
The point (x, y) satisfies $x < y$ if and only if it belongs to the shaded triangle bounded by the lines $x = y$, $y = 1$, and $x = 0$, the area of which is $1/2$. The ratio of the area of the triangle to the area of the rectangle is

$$\frac{1/2}{4} = \frac{1}{8}$$



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Construct eight slots, six to place the cookies in and two to divide the cookies by type. Let the number of chocolate chip cookies be the number of slots to the left of the first divider, the number of oatmeal cookies be the number of slots between the two dividers, and the number of peanut butter cookies be the number of slots to the right of the second divider. For example, □□□ | □□ | □ represents three chocolate chip cookies, two oatmeal cookies, and one peanut butter cookie.

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