Solutions: Math League Contest #5, 2/24/2009
ACHS Math Competition Team

Peter S. Simon

3 March 2009

Upcoming Events

- Mar 24: Math League Contest #6

Problem 5-1

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If the side length of the original square was L, then the perimeter of the rectangle is $3L = 18$. So $L = 6$ and the perimeter of the original square was $4L = 24$.

Problem 5-2

Each day last week, every member of the Jogging Team jogged the same integral number of km as every other team member (and the team has more than one member). The total of the distances jogged by the team members last Monday was 287 km. Last Wednesday, that total was 492 km. How many team members are there?

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Factoring both numbers, we see that

$$287 = 7 \times 41, \quad 492 = 12 \times 41$$

so the team must have 41 members.

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The two three-digit numbers are therefore 698 and 745 and their difference is

$$745 - 698 = 47.$$

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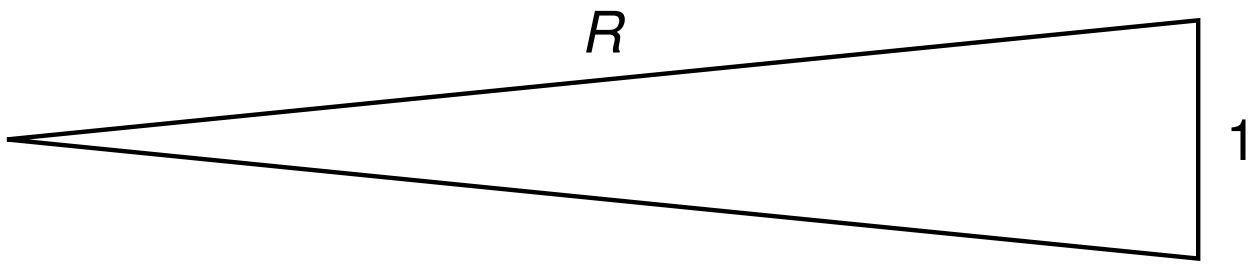

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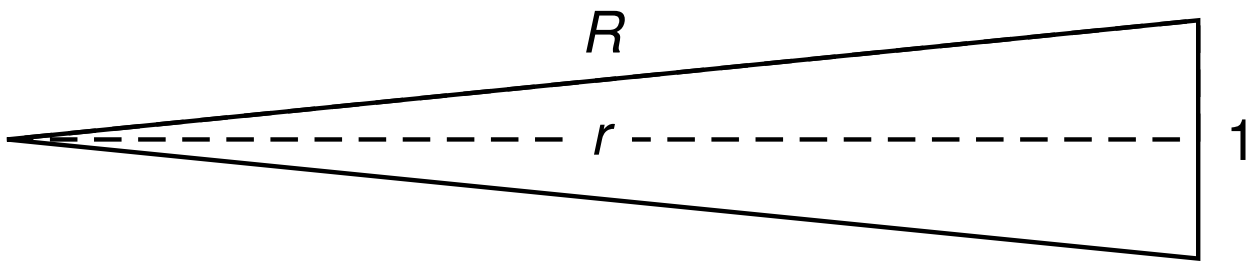

Let r and R be the radii of the inscribed and circumscribed circles, respectively. Since each segment of the polygon has length 1, then from the Pythagorean theorem $R^2 = r^2 + (1/2)^2 = r^2 + 1/4$.

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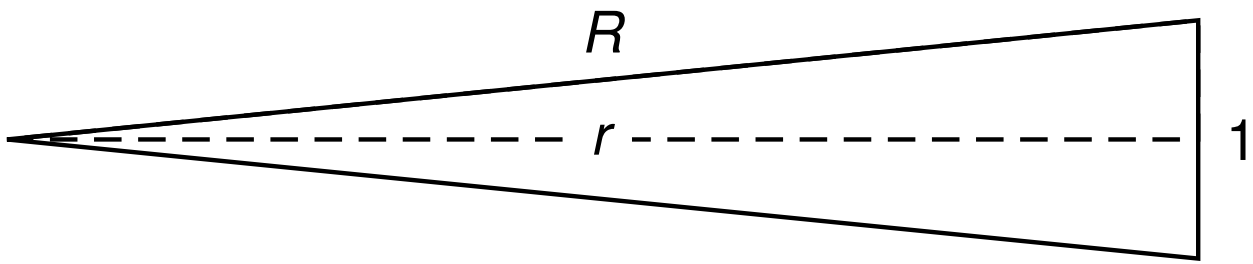

Let r and R be the radii of the inscribed and circumscribed circles, respectively. Since each segment of the polygon has length 1, then from the Pythagorean theorem $R^2 = r^2 + (1/2)^2 = r^2 + 1/4$. It follows that the difference of the areas of the circles is

$$\pi R^2 - \pi r^2 = \pi \left(r^2 + \frac{1}{4} - r^2 \right) = \frac{\pi}{4} \approx 0.7854$$

Problem 5-6

All students at the Academy of Music and Math take both music and math. The probability that a student has an *A* in math is $1/6$. The probability that a student has an *A* in music is $5/12$. The probability that a student with an *A* in math has an *A* in music plus the probability that a student with an *A* in music has an *A* in math is $7/10$. What is the probability that a student has *A*s in both subjects?

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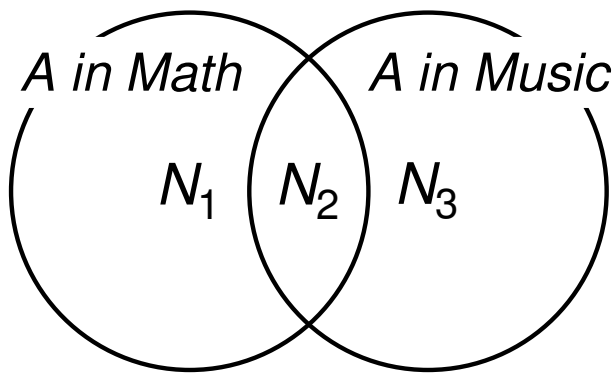
We solve this problem with the aid of a Venn diagram. Let N_1, N_2, and N_3 be the numbers of students with, respectively: an *A* in math but not in music, *A*s in both subjects, and an *A* in music but not in math. The total number of students is $N = N_1 + N_2 + N_3$.



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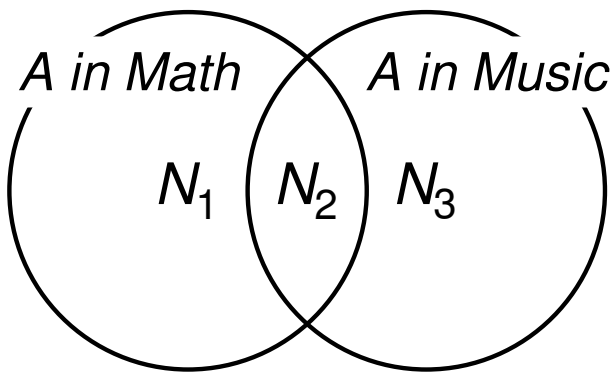
The probability we seek of getting *A*s in both subjects is N_2/N.



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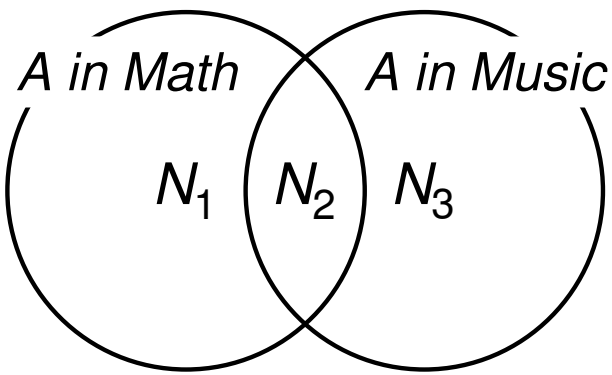

$$\frac{N_1 + N_2}{N} = \frac{1}{6}$$

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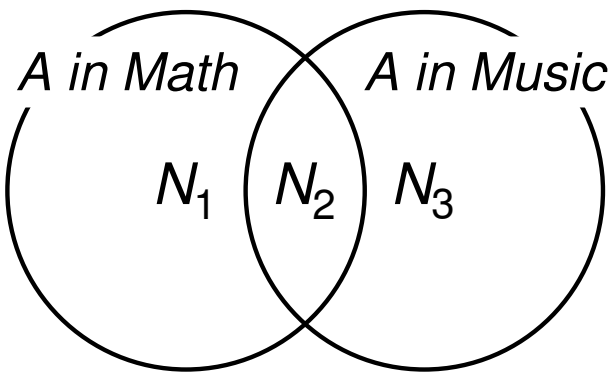

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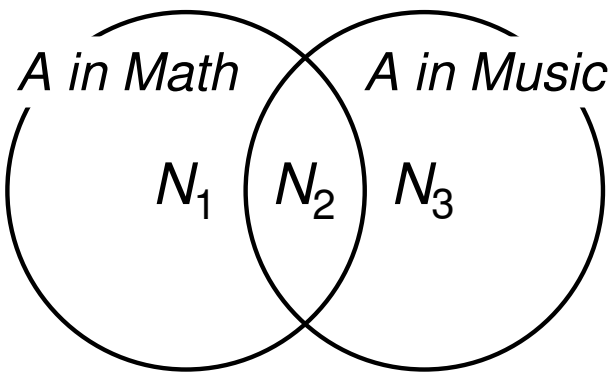

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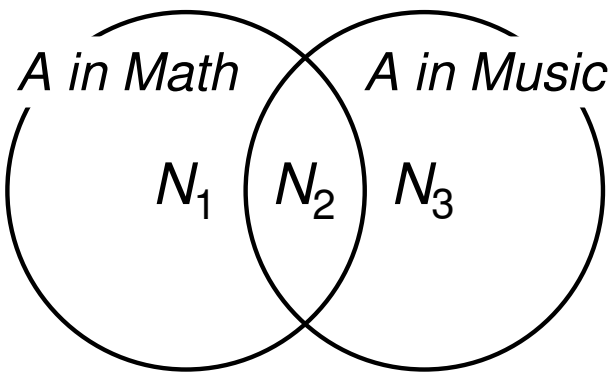

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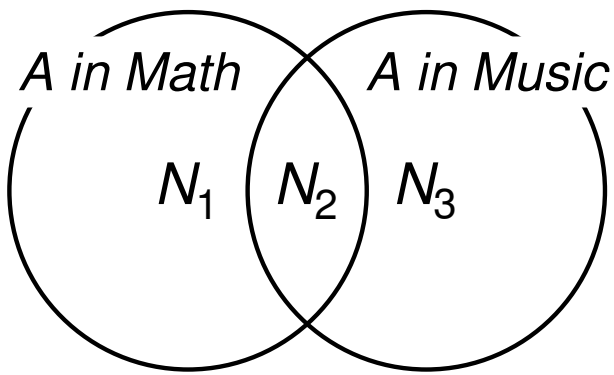

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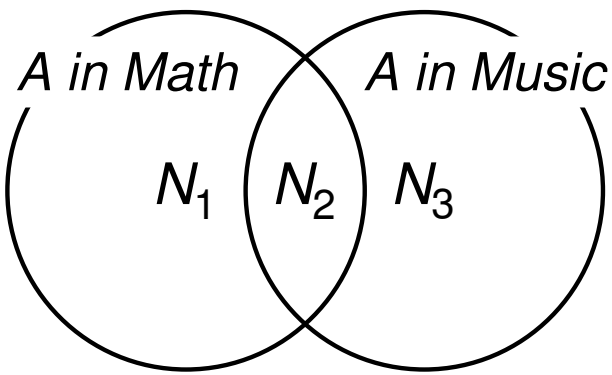

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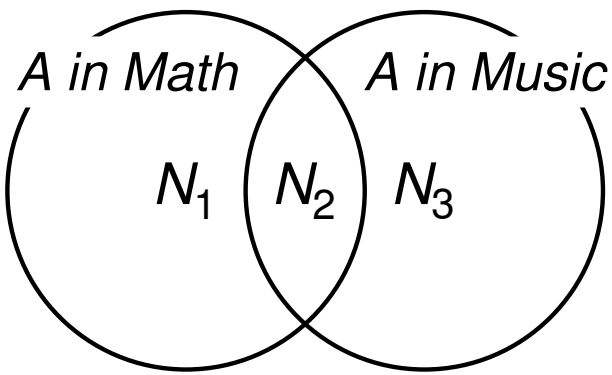

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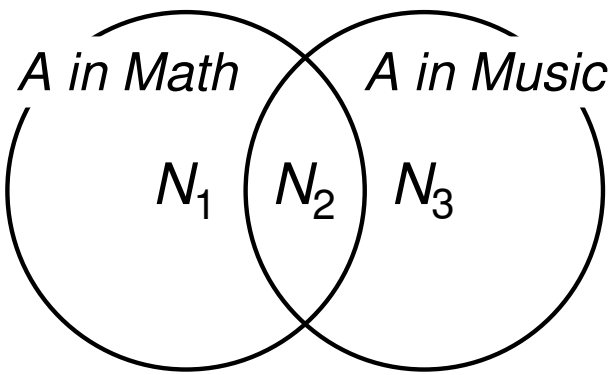

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