Solutions: Math League Contest #3, December 16, 2008

ACHS Math Competition Team

16 Dec 2008

Upcoming Events

- Jan 13: Math League Contest #4
- Feb 10: AMC 10/12 Competitions
- Feb 24: Math League Contest #5
- Mar 24: Math League Contest #6

Problem 3-1

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Since $12/50 = 24/100$, the \$12 discount on the \$50 price amounts to a 24% discount.

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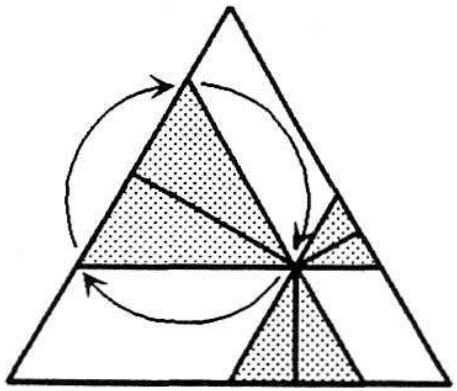
To maximize the product $(a + c)(b + d)$, we choose the 2 factors as nearly equal as possible. In this case, they can both be chosen to be equal to 7:

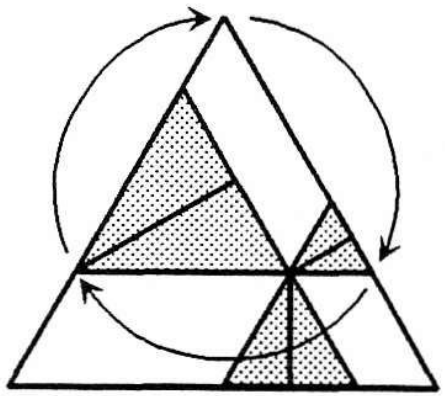
$$(a + c)(b + d) = (2 + 5)(3 + 4) = 49.$$

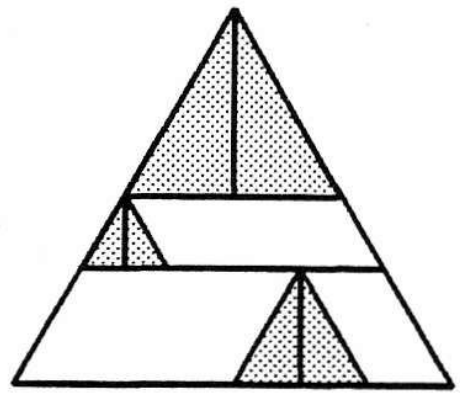
Problem 3-4

What is the perimeter of an equilateral triangle in which the distances from an interior point to the sides are $3\sqrt{3}$, $4\sqrt{3}$, and $5\sqrt{3}$?

(**Note**: A theorem useful in solving this problem is illustrated below.)







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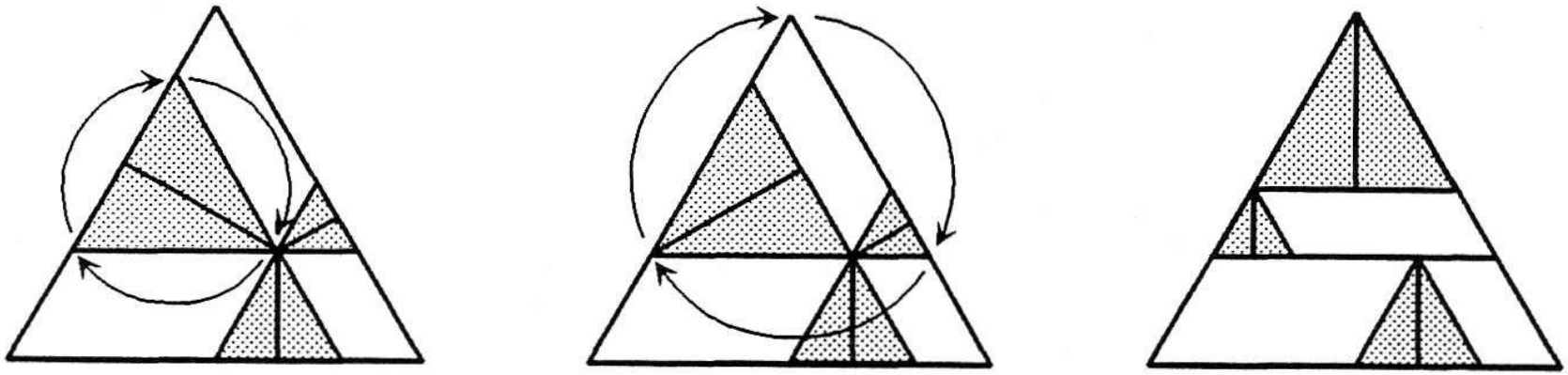

The three pictures are a "Proof Without Words" of Viviani's Theorem: *The sum of the distances from any interior point to the sides of an equilateral triangle is equal to the length of the triangle's alititude*.

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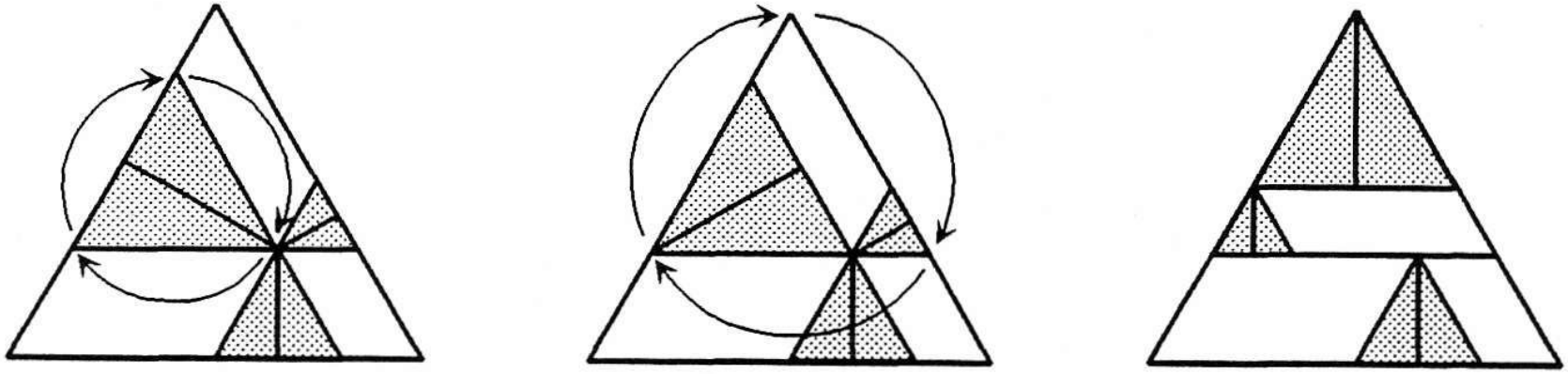

The three pictures are a "Proof Without Words" of Viviani's Theorem: *The sum of the distances from any interior point to the sides of an equilateral triangle is equal to the length of the triangle's alititude*. The sum of the lengths of the 3 altitudes is $12\sqrt{3}$, so the length of one side of the large triangle is 24, and its perimeter measures 72.

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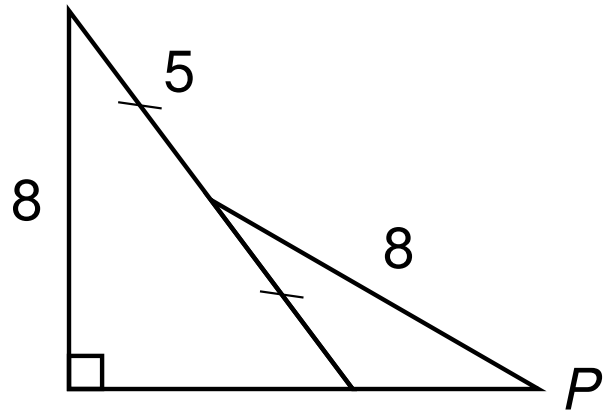
Line segment ℓ connects the midpoint of the hypotenuse of a 6-8-10 right triangle to a point P on the extension of the triangle's shorter leg, as shown. If the length of ℓ is 8, what is $m\angle P$?



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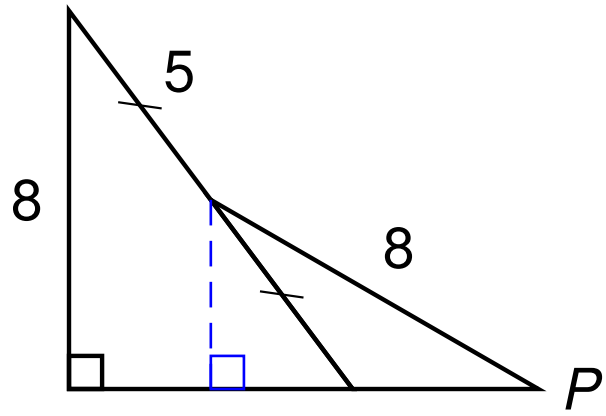

Drop a perpendicular from the midpoint of the hypotenuse to the shorter leg. Since the new segment is parallel to the longer leg and passes through the midpoint of the hypotenuse, its length is half that of the longer leg, or

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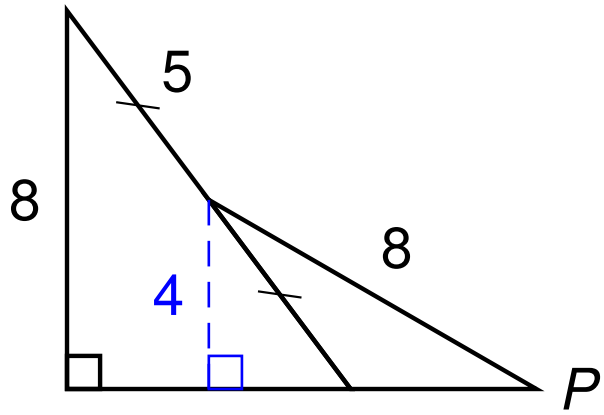

Drop a perpendicular from the midpoint of the hypotenuse to the shorter leg. Since the new segment is parallel to the longer leg and passes through the midpoint of the hypotenuse, its length is half that of the longer leg, or 4 units. We've created a right triangle whose shorter leg and hypotenuse are 4 and 8 units long, respectively. Therefore, $m\angle P = 30^o$.

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