ACHS Math Team
Lecture: Pascal's Triangle and the Binomial Theorem

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Pascal's Triangle and Powers of 11

Let's look at the first few integer powers of 11:

$$
\begin{array}{rccccccccc}
11^0 = & & & & & 1 & & & & \\
11^1 = & & & & 1 & & 1 & & & \\
11^2 = & & & 1 & & 2 & & 1 & & \\
11^3 = & & 1 & & 3 & & 3 & & 1 & \\
11^4 = & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

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11^4 = & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

The decimal digits in these powers of 11 are equal to the following combinations of *n* things *k* at a time:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

What is the connection between Pascal's triangle (consisting of $\binom{n}{k}$, the number of combinations of n things taken k at a time) and the digits occuring in the first few integral powers of 11?

Binomial Theorem and Pascal's Triangle

The connection between Pascal's triangle and the digits occurring in the first few integer powers of 11 is given by a mathematical identity known as the binomial theorem. Pascal's Triangle is named after the French mathematician, Blaise Pascal (1623–1662), although it had been described 5 centuries earlier by the Chinese and also and the by the Persians. The binomial theorem was known for the case $n = 2$ by Euclid ca 300 BC, and stated in modern form by Pascal.

A binomial is just the sum of two numbers: $a + b$.

Binomials Raised to Powers

Consider the binomial

$$P = (a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b).$$

We want to keep track of where the a's and b's in the product come from, so we will temporarily add subscripts to each factor in the product:

$$P = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4)(a_5 + b_5)$$

If we multiply this out, we get terms like $a_1 b_2 a_3 a_4 b_5$. The total number of terms will be 2^5 (why?). These terms result from selecting one of the terms (a_k or b_k) from each factor in the original product. Note that the number of as and bs in each term must add up to 5. How many terms are there with three as and two bs? There must be $\binom{5}{3} = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$ (why?)

The Binomial Theorem

Since there really aren't any subscripts on the as and bs, each of these 10 terms is identical and is equal to a^3b^2. Since there are 10 of them, the product will contain the term $10a^3b^2$.

From these considerations, one can prove that for any numbers a and b and any nonnegative integer n:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots \\ &\qquad \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n.\end{aligned}$$

This is the motivation for referring to the quantity $\binom{n}{k}$ as a binomial coefficient.

Examples for Binomial Theorem

Example ($n = 2$)

$$(a+b)^2 = \sum_{k=0}^{2} \binom{2}{k} a^{2-k} b^k = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2 = a^2 + 2ab + b$$

Example ($n = 2$ with $a = 5$ and $b = 2$)

$$(5+2)^2 = \sum_{k=0}^{2} \binom{2}{k} 5^{2-k} 2^k = \binom{2}{0} 5^2 2^0 + \binom{2}{1} 5^1 2^1 + \binom{2}{2} 5^0 2^2$$

$$= 5^2 + 2 \times 5 \times 2 + 2^2 = 49$$

Binomial Theorem and Pascal's Triangle (Cont.)

The theorem is sometimes very convenient when it is easy to raise a and b to various powers and more difficult to calculate $(a+b)$ raised to the same powers. For example $11 = 10 + 1$ and it is very easy to raise 10 and 1 to any integer powers.

$$\begin{aligned}
11^3 &= (10+1)^3 \\
&= \binom{3}{0}10^3 1^0 + \binom{3}{1}10^2 1^1 + \binom{3}{2}10^1 1^2 + \binom{3}{3}10^0 1^3 \\
&= \binom{3}{0} \times 1000 + \binom{3}{1} \times 100 + \binom{3}{2} \times 10 + \binom{3}{3} \times 1 \\
&= 1 \times 1000 + 3 \times 100 + 3 \times 10 + 1 \times 1 = 1331.
\end{aligned}$$

Note that we can find all of the binomial coefficients for $n = 3$ by looking at the decimal digits in the number 11^3. Similarly, we can find all the binomial coefficients for $n = 2$ by examining the decimal representation of 11^2. But consider how multiplication by 11 is accomplished. . .

Binomial Theorem and Pascal's Triangle (Cont.)

Consider 11^2: $$\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ +11 \\ \hline 121 \end{array}$$ and 11^3: $$\begin{array}{r} 121 \\ \times 11 \\ \hline 121 \\ +121 \\ \hline 1331 \end{array}$$

Each successive power of 11 is obtained by adding two copies of the preceeding power, after shifting one of the factors one place to the left. This observation helps explain the workings of **Pascal's Triangle.**

Pascal's Triangle Begins with Decimal Expansion of $\mathbf{11^n}$

$$
\begin{array}{rccccccccccccc}
11^0 = & & & & & & & 1 & & & & & & \\
11^1 = & & & & & & 1 & & 1 & & & & & \\
11^2 = & & & & & 1 & & 2 & & 1 & & & & \\
11^3 = & & & & 1 & & 3 & & 3 & & 1 & & & \\
11^4 = & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & & \cdots & & & & & &
\end{array}
$$

Pascal's Triangle

We can generalize the interpretation of Pascal's triangle to binomials involving any two numbers *a* and *b*.

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
(a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 \\
&\cdots
\end{aligned}
$$

Pascal's Triangle (Cont.)

$$
\begin{aligned}
(a+b)^0 &= \binom{0}{0} \\
(a+b)^1 &= \binom{1}{0}a + \binom{1}{1}b \\
(a+b)^2 &= \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 \\
(a+b)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \\
(a+b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4 \\
(a+b)^5 &= \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5 \\
(a+b)^6 &= \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6 \\
&\cdots
\end{aligned}
$$

Pascal's Triangle (Cont.)

Retaining only the numeric coefficients we again obtain Pascal's Triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
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Pascal's triangle has the following interesting properties:

1. The first and last number in each row is 1 (since $\binom{n}{0} = \binom{n}{n} = 1$).

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\end{array}
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Pascal's triangle has the following interesting properties:

1. The first and last number in each row is 1 (since $\binom{n}{0} = \binom{n}{n} = 1$).
2. Every other number in the array can be obtained by adding the two numbers directly above it (since $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$).

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3. The triangle has left/right mirror symmetry (since $\binom{n}{k} = \binom{n}{n-k}$).

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3. The triangle has left/right mirror symmetry (since $\binom{n}{k} = \binom{n}{n-k}$).
4. The entries in each row sum to twice the value of the previous row. Row n sums to the value 2^n (counting rows starting from 0).

Pascal's Triangle Example Problem 1

Circle the coefficient corresponding to $\binom{6}{3}$ in Pascal's Triangle, below. Verify using the formula for combinations that the value shown is correct.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
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 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 5 \times 4 = \boxed{20}$$

Pascal's Triangle Example Problem 2

Calculate the following quantity without using pencil and paper or calculator:

$$99^3 + 3 \times 99^2 + 3 \times 99 + 1 = ?$$

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Calculate the following quantity without using pencil and paper or calculator:

$$99^3 + 3 \times 99^2 + 3 \times 99 + 1 = ?$$

Answer: 1,000,000. Note that

$$\begin{aligned} 99^3 + 3 \times 99^2 + 3 \times 99 + 1 &= 99^3 1^0 + 3 \times 99^2 1^1 + 3 \times 99^1 1^2 + 99^0 1^3 \\ &= (99+1)^3 \\ &= 100^3 = (10^2)^3 = 10^6. \end{aligned}$$

Homework Problem for Binomial Theorem

Derive a simple expression for the number of subsets that can be found for a set containing n elements.

Hint: Add up the number of subsets having 0 elements, 1 element, 2 elements, . . . , n elements. (Recall that the empty set (the set with 0 elements) is considered a subset of every set.)