2009 Fall Startup Event
Solutions

1. Just do the arithmetic: $352+184=536$.

2. Just do the arithmetic: $552\div 8=69$.

3. $\frac{5}{6}\times\frac{10}{21}=\frac{5}{3}\times\frac{5}{21}=\frac{25}{63}$

4. $\frac{48\times 100}{160}=\frac{48\times 10}{16}=3\times 10=30$

5. A meter is 100 cm, so 3.5 meters is 350 cm.

6. Use PEMDAS: $2(4-6)^3+8=2(-2)^3+8=2(-8)+8=-16+8=-8$.

7. $\sqrt{252}=\sqrt{4\times 63}=2\sqrt{9\times 7}=2\times 3\sqrt{7}=6\sqrt{7}$

8. $\frac{84}{4+\sqrt{2}}\times\frac{4-\sqrt{2}}{4-\sqrt{2}}=\frac{84\left(4-\sqrt{2}\right)}{16-2}=\frac{84\left(4-\sqrt{2}\right)}{14}=6\left(4-\sqrt{2}\right)=24-6\sqrt{2}$

9. $19^2-15^2=(19-15)(19+15)=4\cdot 34=136$

10. The tenths digit is just after the decimal point, so the answer is 6.

11. $6z+11=113$ becomes $6z=102$, giving $z=17$.

12. $4(y+7)-5=3(2y-9)$ becomes $4y+28-5=6y-27$, then $50=2y$, giving $y=25$.

13. $3x^2-14x-24=0$ factors to $(3x+4)(x-6)=0$, with solutions of $-\frac{4}{3}$ and 6.

14. Twice $3w+v=14$ plus $w-2v=7$ gives $7w=35$, so that $w=5$, giving $v=-1$.

15. This problem is all about **rates**. James' is $\frac{1}{3}$ and John's is $\frac{1}{2}$, for a combined rate of $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$ of the dishes per hour. Thus, to do 1 set of dishes will take $\frac{6}{5}$ hours, which is $\frac{6}{5}\times 60=6\times 12=72$ minutes.

16. $\frac{1200}{48}=\frac{100}{4}=25$

17. The two numbers center about 43.5, and are each 9.5 away from it, so the smaller one is $43.5-9.5=34$

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18. The slope is $\frac{-1-(-7)}{-2-1}=\frac{-1+7}{-3}=\frac{6}{-3}=-2$, so the line is $y=-2x+b$. Substituting the first point gives $-7=-2\cdot 1+b$, then $-5=b$, for an answer of $y=-2x-5$.

19. The slope of the line $6x+4y=21$ is $-\frac{A}{B}=-\frac{6}{4}=-\frac{3}{2}$, so the perpendicular slope is the negative inverse of this, $\frac{2}{3}$.

20. Use the memorized formula: $\frac{|4\cdot 4-3(-6)-5|}{\sqrt{4^2+(-3)^2}}=\frac{29}{5}$.

21. The slope of $6x-3y=11$ is $-\frac{A}{B}=-\frac{6}{-3}=2$, so any line of the form $y=2x+b$ can be an answer except for $b=-\frac{11}{3}$.

22. The vertex lies on the axis of symmetry $x=-\frac{b}{2a}=-\frac{-48}{2\cdot 4}=6$, so that $y=4\cdot 6^2-48\cdot 6+17=144-288+17=-127$.

23. Forks cost $\frac{5N}{4}$ cents each, so F forks cost $\frac{5NF}{4}$ cents, which is $\frac{5NF}{4\cdot 10}=\frac{NF}{4\cdot 2}=\frac{NF}{8}$ dimes.

24. If all 42 coins were dimes, they'd be worth \$4.20. We actually have twenty cents less, so we need to turn 4 dimes into nickels (each change loses five cents).

25. From three to six is three hours, so 180 minutes. Adding 15 and 10 gives 205.

26. Cross-multiplying gives $(u+21)(u-7)=(u+15)(u-9)$, which becomes $u^2+14u-147=u^2+6u-135$, then $8u=12$, resulting in $u=\frac{3}{2}$.

27. $(2t-3)(t+2)(3t+4)=(2t^2+t-6)(3t+4)=6t^3+11t^2-14t-24$ for an answer of 11.

28. $(-3,8)$ is in the upper-left quadrant, which is the second quadrant going counter-clockwise from the upper-right quadrant.

29. The *x* and *y*-intercepts of $2x+4y=20$ are $(10,0)$ and $(0,5)$, for a distance of $\sqrt{10^2+5^2}=5\sqrt{2^2+1^2}=5\sqrt{4+1}=5\sqrt{5}$.

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30. Reflecting across the line $y=-x$ switches the coordinates and the signs, for an answer of $(-17,-5)$.

31. The Pythagorean Theorem gives $\sqrt{8^2-6^2}=2\sqrt{4^2-3^2}=2\sqrt{16-9}=2\sqrt{7}$.

32. $A=\dfrac{10^2\sqrt{3}}{4}=\dfrac{100\sqrt{3}}{4}=25\sqrt{3}$

33. Hopefully, you have “isosceles” memorized.

34. The third side could be as short as $73-37+1=37$ and as long as $73+37-1=109$, for $109-37+1=110-37=73$ possibilities.

35. The perimeter is twice the sum of the length and width: $2(14+11)=2\times 25=50$.

36. A $60°$ sector is one-sixth of a circle, so its area is $\dfrac{1}{6}\times 6^2\pi=6\pi$.

37. The pentagon has five sides, so it’s perimeter is $5\cdot 22=110$.

38. The *exterior* angle is $360\div 15=24$, so the interior angle is $180-24=156$.

39. The 12 and 24 sides are in the ratio of 1:2, so they must correspond to the sides of 3 and 6, so that every dimension of the larger shape is four times that of the smaller shape, and its area is $4^2=16$ times that of the smaller shape, for an answer of $16\times 2=32$.

40. $V=\dfrac{1}{3}\pi r^2 h=\dfrac{1}{3}\pi\cdot 9^2\cdot 6=\pi\cdot 81\cdot 2=162\pi$

41. An octahedron has eight faces, each of which is an equilateral triangle, so that it has just six vertices (each of which can be the center of a face of a cube).

42. $94=\dfrac{113+r}{2}$ becomes $188=113+r$, then $r=75$.

43. $A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{10\cdot 5\cdot 3\cdot 2}=5\cdot 2\sqrt{3}=10\sqrt{3}$, where *s* is the semi-perimeter.

44. Drawing the altitudes of the triangle, one can determine that the radius of the inscribed circle would be 6, so that the radius of the desired circle is $\dfrac{12-6}{2}=3$, for an area of 9π.

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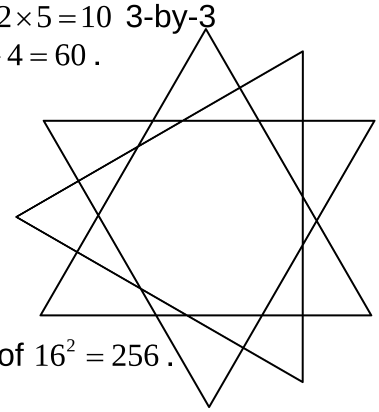
45. Because of the similar right triangles created, the length of the tangent is essentially the remaining leg of a right triangle with a hypotenuse of 36 and a leg of $12+18=30$, for an answer of $\sqrt{36^2-30^2}=6\sqrt{6^2-5^2}=6\sqrt{36-25}=6\sqrt{11}$.

46. There are $4\times 7=28$ 1-by-1 squares, $3\times 6=18$ 2-by-2 squares, $2\times 5=10$ 3-by-3 squares, and $1\times 4=4$ 4-by-4 squares, for a total of $28+18+10+4=60$.



47. $\frac{n(n-3)}{2}=\frac{16\cdot 13}{2}=8\cdot 13=104$

48. There are 20 regions in the figure to the right:
1 exterior, 1 central, 9 “tips” and 9 “interior”.

49. If the circle’s radius is 8, the square’s sides are 16, for an area of $16^2=256$.

50. $SA=2(9\cdot 6+9\cdot 13+6\cdot 13)=2(54+117+78)=2\cdot 249=498$

51. $(3-5i)(i+2)=3i+6-5i^2-10i=-7i+6-5(-1)=-7i+6+5=11-7i$

52. Completing the square, $2x^2-5y^2+8x+30y=101$ becomes $2(x+2)^2-5(y-3)^2=101+2\cdot 2^2-5(-3)^2$, so that the center is $(-2,3)$.

53. $y=\frac{1}{x+2}$ is a rectangular hyperbola with asymptotes of $y=0$ and $x=-2$, while $y=5-|x-3|$ is an upside-down V with a vertex at $(3,5)$, so that they cross at one point to the right of $(3,5)$ and above the x-axis, one point below $(3,5)$ and to the right of the y-axis, and one point to the left of the y-axis and below the x-axis.

54. The powers of 2 are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, and 1024. Clearly, if $q=5$, $2^{2q}=1024$, so we’d have met the criteria. But, if $q=4$ the function falls far below the desired value of 1000, so the answer is 5.

55. $41=\frac{1}{3}p+50$ becomes $-9=\frac{1}{3}p$, so that $p=-27$.

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56. For $m > -1$, $k(m) = \dfrac{3m+6}{|m+1|+1}$ is really $k(m) = \dfrac{3m+6}{m+2} = 3$. For $m < -1$, $k(m) = \dfrac{3m+6}{|m+1|+1}$ is really $k(m) = \dfrac{3m+6}{-m} = -3 + \dfrac{6}{-m}$. For m near –1, $k(m)$ gets near $-3 + \dfrac{6}{-(-1)} = -3+6 = 3$, so the function is continuous there. For very negative values of m, $k(m) = -3 + \dfrac{6}{-m} \approx -3 + \dfrac{6}{-(-\infty)} \approx -3+\delta$, so the range is $(-3,3]$.

57. The relevant equation is $T = Pe^{rt} = 3000e^{.05\cdot 20} = 3000e$.

58. The product of the roots of a polynomial is $(-1)^n \dfrac{z}{a} = (-1)^3 \dfrac{6}{3} = -1\times 2 = -2$.

59. The h^3 term involves choosing 2h three times and –1 twice, so is $5c3 \cdot 2^3 (-1)^2 = 10\cdot 8 = 80$.

60. $\log_{16} 512 = \dfrac{\ln 512}{\ln 16} = \dfrac{9\ln 2}{4\ln 2} = \dfrac{9}{4}$

61. The digits of a base six number represent 1's, 6's, 36's, 216's, etc. 235 has 1 216, leaving 19. Thus, there are no 36's, 3 6's, and 1 1, for an answer of 1031.

62. $840 = 10\cdot 7\cdot 12 = 2^3\cdot 3\cdot 5\cdot 7$

63. $108 = 4\cdot 27 = 2^2\cdot 3^3$, so the sum of the factors is $(1+2+4)(1+3+9+27) = 7\cdot 40 = 280$.

64. $135 = 3\cdot 45 = 3^3\cdot 5$, while $72 = 8\cdot 9 = 2^3\cdot 3^2$, so the LCM will be $2^3\cdot 3^3\cdot 5 = 8\cdot 135 = 1080$.

65. A number is divisible by 9 if the sum of its digits is divisible by 9. 82G459 has a digital sum of $28+G$, so G must be 8.

66. 54 is divisible by 2, as are all even numbers. 55 is divisible by 5 and 57 is divisible by 3, so 59 is the smallest prime greater than 53.

67. There are $6\cdot 6 = 36$ ways to roll two dice, and there are three ways to roll a sum of 10 (4&6 or 5&5 or 6&4), for a probability of $\dfrac{3}{36} = \dfrac{1}{12}$.

68. There are $8c3 = \dfrac{8\cdot 7\cdot 6}{3\cdot 2} = 8\cdot 7 = 56$ ways to choose three tiles. There are $3c2 = 3$ ways to choose two blues and $5c1 = 5$ ways to choose a red, for an answer of $\dfrac{5\cdot 3}{56} = \dfrac{15}{56}$.

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69. Because all spaces are taken, this is essentially just how many ways can we choose three spots out of the eight for the whites. If the first two are adjacent, the third could be in one of five relative locations. If the first two are one apart, there are two possible locations. If the first two are three apart, there are no new locations for the third, so the answer is $5+2=7$.

70. 37 like both, and 1 likes neither. $94-73-1=20$ like Mr. Brown only, so there are $37+20=57$ who like Mr. Brown.

71. The probability that Tom wins is $\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}+\cdots$, which becomes $\frac{1}{6}+\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^3+\cdots$, a geometric sequence with a sum of $\frac{\frac{1}{6}}{1-\frac{1}{6}}=\frac{\frac{1}{6}}{\frac{5}{6}}=\frac{1}{5}$.

72. There is a $\frac{1}{2}$ chance of winning the \$7, so your expected payback is \$3.50. Because you initially spend \$5, your expected gain is -\$1.50.

73. $9\cdot 5^4=9\cdot 625=5625$

74. This is two interspersed sequences, the critical one being 12, 15, 18, 21, __ with a next term of 24.

75. $\sum_{f=2}^{7}\frac{1}{f^2-1}=\sum_{f=2}^{7}\frac{1}{2}\left(\frac{1}{f-1}-\frac{1}{f+1}\right)=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{7}-\frac{1}{8}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{15}{56}\right)=\frac{1}{2}\left(\frac{69}{56}\right)=\frac{69}{112}$

76. The first term is 21 and the 12th term is $21+11(-4)=21-44=-23$, so that there are six pairs that sum to –2, for a total of –12.

77. The sum of the n smallest odds is $n^2=17^2=289$.

78. The sum of the n smallest squares is $\frac{n(n+1)(2n+1)}{6}=\frac{13(14)(27)}{6}=13\cdot 7\cdot 9=819$.

79. If left is forward, we know that there is AD or DA, E…C, EB, and DC. These can only be combined as EBADC, so that Annie is in the middle.

80. A cannot be 5 or higher, as then the sum would have three digits instead of two. If A is 4, then B would be 8 or 9. B cannot be 8, as then B+B would carry, making the B beneath A+A a 9. But B *can* be 9, making our answer 498.

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81. If we write $x=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$, it becomes $x=\sqrt{2+x}$, and squaring both sides gives $x^2=2+x$. This becomes $x^2-x-2=0$ which factors to $(x-2)(x+1)=0$ with solutions of 2 and –1. Because x must be positive, only 2 is our answer.

82. Evaluating each set of four numbers in the pattern gives $2+10+18+\cdots+194$, which is an arithmetic series with 25 terms and thus has a sum of $\frac{25(2+194)}{2}=25\cdot 98=49\cdot 50=2450$.

83. $9\times 8\div 6=12$, as would several permutations.

84. Rewriting the set in ascending order gives 4, 7, 8, 11, 11. We want the median, which is the middle, which is 8.

85. The data set looks something like x-2, x-1, x, 60, 60, with a mean of 48, so $3x-3+120=5\cdot 48$. This becomes $3x=240-117=123$, so that $x=41$.

86. $\begin{bmatrix}1 & -2 & 3\end{bmatrix}\begin{bmatrix}-2 & 1\\ 0 & -1\\ 2 & -3\end{bmatrix}=\begin{bmatrix}1(-2)+(-2)\cdot 0+3\cdot 2 & 1\cdot 1+(-2)(-1)+3(-3)\end{bmatrix}=\begin{bmatrix}-2+6 & 1+2-9\end{bmatrix}=\begin{bmatrix}4 & -6\end{bmatrix}$

87. Drawing a 6x7 rectangle around the triangle, the triangle's area is 42 minus the area of three right triangles: 3x4, 6x3, and 3x7. $42-\frac{1}{2}(12+18+21)=42-\frac{51}{2}=\frac{33}{2}$.

88. $D\cap E'$ is all the numbers between 24 and 89 that are multiples of 5 but not 3, which is 25, 35, 40, 50, 55, 65, 70, 80, and 85, which is 9 elements.

89. $\sin(F+G)=\sin F\cos G+\sin G\cos F=\frac{12}{13}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{5}{13}=\frac{48+15}{65}=\frac{63}{65}$

90. We're looking for the angle between 0 and π that has a cosine of $-\frac{\sqrt{2}}{2}$, which is $\frac{3\pi}{4}$.

91. Spherical coordinates are the distance from the origin, the angle from the positive z-axis, and the angle in the x-y plane from the positive x-axis. Cylindrical coordinates are the radius in the x-y plane, the angle in the x-y plane from the positive x-axis, and the height, z. So, the cylindrical coordinates are $\left(8\sin\frac{\pi}{6},\frac{\pi}{4},8\cos\frac{\pi}{6}\right)=\left(4,\frac{\pi}{4},4\sqrt{3}\right)$.

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92. This is a right triangle, and the smallest angle will be opposite the 5, so that it's tangent is $\frac{5}{12}$.

93. $100° \cdot \frac{\pi}{180°} = \frac{10\pi}{18} = \frac{5\pi}{9}$

94. $A = \frac{1}{2} ab \sin C = \frac{1}{2} \cdot 12 \cdot 18 \sin 120 = 6 \cdot 18 \cdot \frac{\sqrt{3}}{2} = 3 \cdot 18\sqrt{3} = 54\sqrt{3}$

95. $k'(m) = 4(2m-3)^3 \cdot 2 = 8(2m-3)^3$, so $k'(0) = 8(-3)^3 = 8(-27) = -216$.

96. Similar triangles tell us that at the moment in question, the water has a surface area of 36π, so the rate at which it's falling is going to be $\frac{100}{36\pi} = \frac{25}{9\pi}$.

97. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 3\sin(2x)\,dx = -\frac{3}{2}\cos(2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\frac{3}{2}\left(\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{3}\right)\right) = -\frac{3}{2}\left(-\frac{1}{2}\right) = \frac{3}{4}$

98. $2\int_0^2 (16 - x^4)\,dx = 2\left(16x - \frac{1}{5}x^5\right)\Big|_0^2 = 2\left(16(2-0) - \frac{1}{5}(2^5 - 0^5)\right) = 2\left(32 - \frac{32}{5}\right) = \frac{256}{5}$

99. The derivative at the point in question is $y'(-3) = 3(-3)^2 = 27$, and the value at the point in question is $y(-3) = (-3)^3 = -27$, so that the line is $y = 27x + 54$.

100. $\frac{\int_1^5 (x^2 - 2)\,dx}{5-1} = \frac{1}{4}\left(\frac{1}{3}x^3 - 2x\right)\Big|_1^5 = \frac{1}{4}\left(\frac{1}{3}(125-1) - 2(5-1)\right) = \frac{1}{4}\left(\frac{124}{3} - 8\right) = \frac{25}{3}$