ACHS Math Team
Solutions to Math League Contest 1 of 24 Oct 2006
Peter S. Simon

Upcoming Contests

Tues, January 9: Math League Contest 3
Tues, January 30: Math League Contest 4
Tues, February 6: American Mathematics Contest 10/12 (Contest A)
Wed, February 20: American Mathematics Contest 10/12 (Contest B)

AMC 10 participants must be in grades 9 or 10 and be under 17.5 years old on the day of the contest.
AMC 12 participants must be in high school and less than 19.5 years old on the day of the contest.
Both tests are multiple choice and last 75 minutes.

Problem 1-1

Which positive integer n satisfies $n^{2006} + 2n^{2007} = 3$?

Problem 1-1

Which positive integer n satisfies $n^{2006} + 2n^{2007} = 3$?

If $n > 1$ then the left member is greater than 3. Since n must be a positive integer, we see by inspection that $n = 1$.

Problem 1-2

In non-equilateral isosceles triangle T, the length of each side is an integer. What is the least possible perimeter of T?

Problem 1-2

In non-equilateral isosceles triangle T, the length of each side is an integer. What is the least possible perimeter of T?

We see it is not possible for the two congruent legs to be of length 1, because the base's length would have to be at least 2 (since it has integer length and is not congruent to the legs). Therefore, try setting the leg lengths to 2. Then the minimum base length is 1 and the minimum possible perimeter is

$$\text{Minimum perimeter} = 2 + 2 + 1 = 5.$$

Problem 1-3

Buses *A* and *B* always arrive on time: *A* every 16 minutes and *B* every 9 minutes. I take 6 minutes to walk to *A*'s stop and 10 minutes to walk to *B*'s. I leave at random times, and I always walk to the same stop. To which stop (*A* or *B*) should I walk to minimize the *expected* amount of time I'd take both to walk to that stop and to wait for the next bus to arrive?

Problem 1-3

Buses *A* and *B* always arrive on time: *A* every 16 minutes and *B* every 9 minutes. I take 6 minutes to walk to *A*'s stop and 10 minutes to walk to *B*'s. I leave at random times, and I always walk to the same stop. To which stop (*A* or *B*) should I walk to minimize the *expected* amount of time I'd take both to walk to that stop and to wait for the next bus to arrive?

Since I leave at random times, my wait for bus *A* is between 0 and 16 minutes, so my waiting time for bus *A* averages 8 minutes.

Problem 1-3

Buses *A* and *B* always arrive on time: *A* every 16 minutes and *B* every 9 minutes. I take 6 minutes to walk to *A*'s stop and 10 minutes to walk to *B*'s. I leave at random times, and I always walk to the same stop. To which stop (*A* or *B*) should I walk to minimize the *expected* amount of time I'd take both to walk to that stop and to wait for the next bus to arrive?

Since I leave at random times, my wait for bus *A* is between 0 and 16 minutes, so my waiting time for bus *A* averages 8 minutes. Similary, my wait for bus *B* is between 0 and 9 minutes, so it averages 4.5 minutes.

Problem 1-3

Buses *A* and *B* always arrive on time: *A* every 16 minutes and *B* every 9 minutes. I take 6 minutes to walk to *A*'s stop and 10 minutes to walk to *B*'s. I leave at random times, and I always walk to the same stop. To which stop (*A* or *B*) should I walk to minimize the *expected* amount of time I'd take both to walk to that stop and to wait for the next bus to arrive?

Since I leave at random times, my wait for bus *A* is between 0 and 16 minutes, so my waiting time for bus *A* averages 8 minutes. Similary, my wait for bus *B* is between 0 and 9 minutes, so it averages 4.5 minutes. For either bus, the expected amount of total elapsed time is the sum of the walk time and the average wait time:

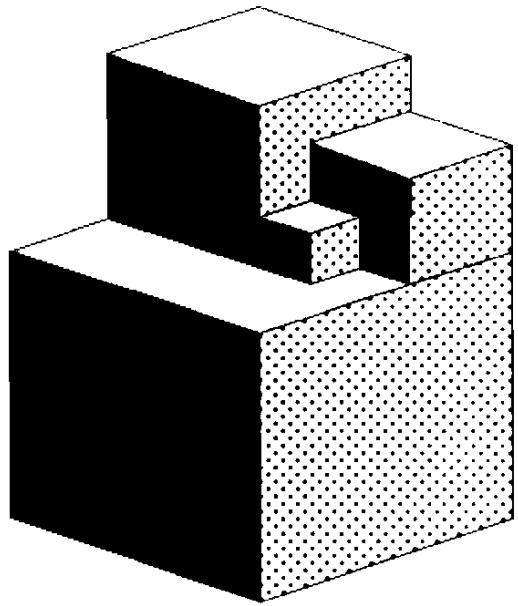
$$\text{Bus A expected total elapsed time} = 6 + 8 = 14\,\text{min}$$

$$\text{Bus B expected total elapsed time} = 10 + 4.5 = 14.5\,\text{min}$$

For the shortest total expected elapsed time, use Bus *A*

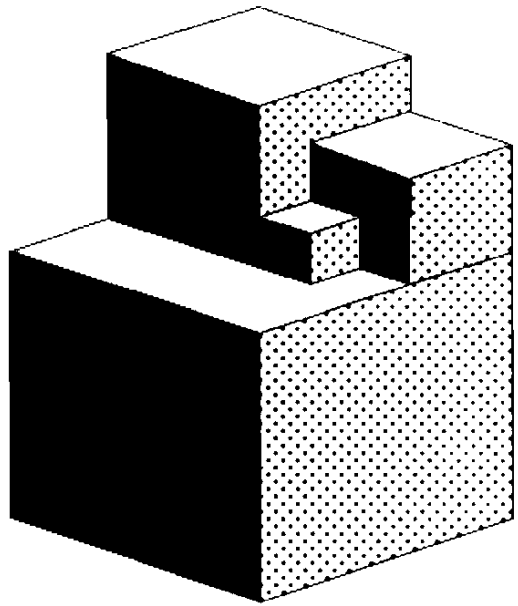
Problem 1-4

As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



Problem 1-4

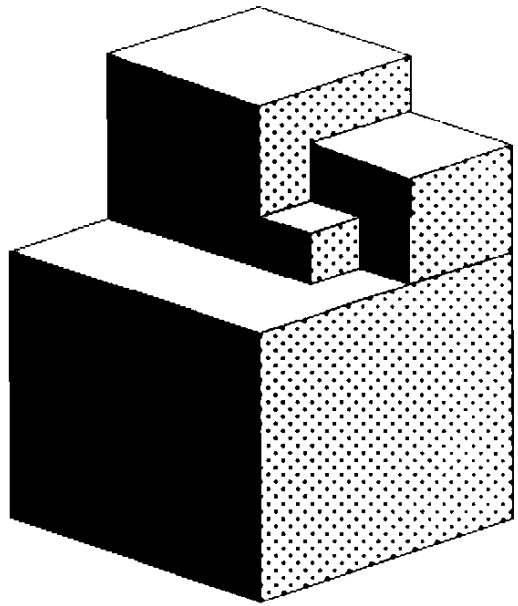
As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5.

Problem 1-4

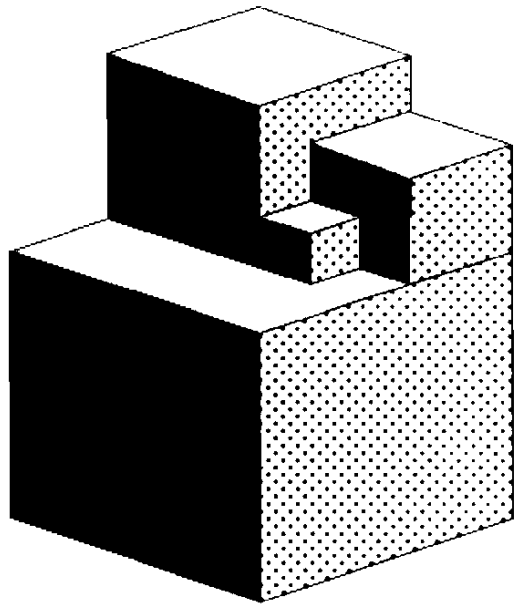
As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5. Each cube has six faces. The areas of the faces are 1, 4, 9, and 25.

Problem 1-4

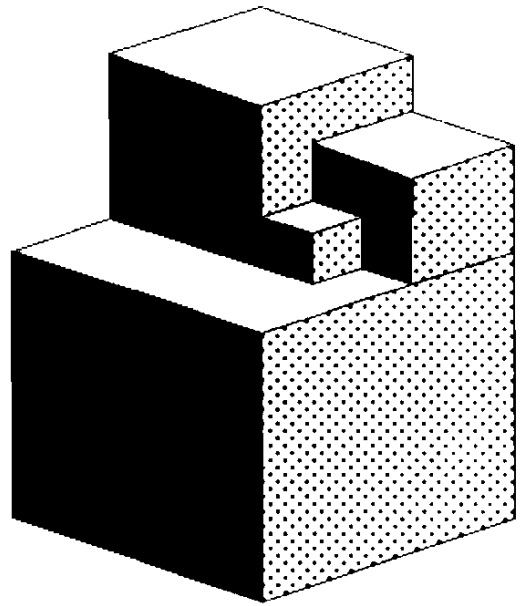
As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5. Each cube has six faces. The areas of the faces are 1, 4, 9, and 25. If every face were visible, the total surface area would be $6(1 + 4 + 9 + 25) = 234$, an upper bound on the possible answer.

Problem 1-4

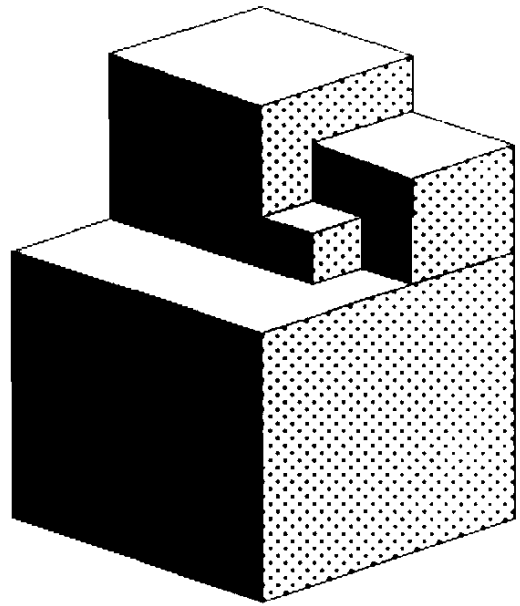
As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5. Each cube has six faces. The areas of the faces are 1, 4, 9, and 25. If every face were visible, the total surface area would be $6(1 + 4 + 9 + 25) = 234$, an upper bound on the possible answer. The (white) top and bottom both have surface area 25.

Problem 1-4

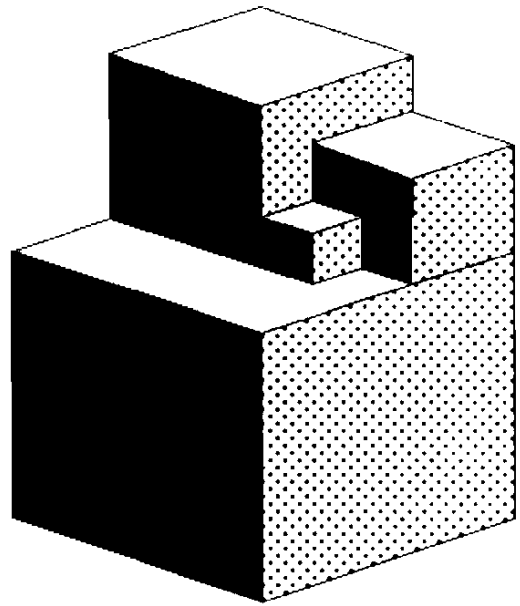
As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5. Each cube has six faces. The areas of the faces are 1, 4, 9, and 25. If every face were visible, the total surface area would be $6(1 + 4 + 9 + 25) = 234$, an upper bound on the possible answer. The (white) top and bottom both have surface area 25. The (dotted) right and left both have surface area $25 + 9 = 34$, and the (black) front and back both have surface area $25 + 9 + 4 = 38$.

Problem 1-4

As shown, four cubes (with respective volumes 1, 8, 27, and 125) are attached to each other in a way that minimizes the total surface area of the resulting configuration. What is the total surface area of this configuration?



The edge lengths of the four cubes are 1, 2, 3, and 5. Each cube has six faces. The areas of the faces are 1, 4, 9, and 25. If every face were visible, the total surface area would be $6(1 + 4 + 9 + 25) = 234$, an upper bound on the possible answer. The (white) top and bottom both have surface area 25. The (dotted) right and left both have surface area $25 + 9 = 34$, and the (black) front and back both have surface area $25 + 9 + 4 = 38$.

$$\text{Total Surf. Area} = 2(25 + 34 + 38) = 194$$

Problem 1-5

The equation $53 = (8 \times 5) + (8 + 5)$ shows how to represent shows how to represent 53 as the product plus the sum of the same two positive integers. What is the least integer greater than 53 which *cannot* be represented this way?

Problem 1-5

The equation $53 = (8 \times 5) + (8 + 5)$ shows how to represent shows how to represent 53 as the product plus the sum of the same two positive integers. What is the least integer greater than 53 which *cannot* be represented this way?

Let our integer be $n = ab + a + b = a(b + 1) + b$.

Problem 1-5

The equation $53 = (8 \times 5) + (8 + 5)$ shows how to represent shows how to represent 53 as the product plus the sum of the same two positive integers. What is the least integer greater than 53 which *cannot* be represented this way?

Let our integer be $n = ab + a + b = a(b + 1) + b$. Although $a(b + 1) + 1$ can not be factored as it is, if I add 1 I can write it as a product:

$$n + 1 = a(b + 1) + b + 1 = (a + 1)(b + 1)$$

Problem 1-5

The equation $53 = (8 \times 5) + (8 + 5)$ shows how to represent shows how to represent 53 as the product plus the sum of the same two positive integers. What is the least integer greater than 53 which *cannot* be represented this way?

Let our integer be $n = ab + a + b = a(b + 1) + b$. Although $a(b + 1) + 1$ can not be factored as it is, if I add 1 I can write it as a product:

$$n + 1 = a(b + 1) + b + 1 = (a + 1)(b + 1)$$

Conclusion

$n + 1$ can be written as a product of positive integers both greater than 1, if and only if n can be represented as the product plus the sum of the two preceding integers:

$$n + 1 = (a + 1)(b + 1) \implies n = ab + (a + b)$$

Problem 1-5, Continued

Example: $n = 54$

$$n + 1 = 55 = 5 \times 11 = (4 + 1)(10 + 1) = (4 \times 10) + (4 + 10) + 1$$

so $n = 54$ can be represented in the required way.

The first time that a number can not be represented in this manner is when the next number is prime. The first prime greater than 53 is 59, so the first number not representable in the required manner is 58.

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$

$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

$x^x y^y = 2^{12x+8y}3^{6x+8y}$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

$$x^x y^y = 2^{12x+8y}3^{6x+8y} = 2^{11\times 2^{11}3^7}3^{7\times 2^{11}3^7}$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

$$x^x y^y = 2^{12x+8y}3^{6x+8y} = 2^{11 \times 2^{11}3^7} 3^{7 \times 2^{11}3^7} = \left(2^{11}\right)^{2^{11}3^7} \left(3^7\right)^{2^{11}3^7}$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

$$x^x y^y = 2^{12x+8y}3^{6x+8y} = 2^{11 \times 2^{11}3^7} 3^{7 \times 2^{11}3^7} = \left(2^{11}\right)^{2^{11}3^7} \left(3^7\right)^{2^{11}3^7} = \left(2^{11}3^7\right)^{2^{11}3^7}$$

Problem 1-6

If $x = 2^{12} \times 3^6$ and $y = 2^8 \times 3^8$, what integer z satisfies $x^x y^y = z^z$? (Note: write answer as a product of powers of primes)

$$x^x y^y = (2^{12}3^6)^x (2^8 \times 3^8)^y = (2^{12x}3^{6x})(2^{8y} \times 3^{8y}) = 2^{12x+8y}3^{6x+8y}$$

$$12x + 8y = 12(2^{12}3^6 + 8(2^8 3^8) = 2^{14}3^7 + 2^{11}3^8$$
$$6x + 8y = 6(2^{12}3^6) + 8(2^8 3^8) = 2^{13}3^7 + 2^{11}3^8$$

Both expressions contain the common factor $2^{11}3^7 \ldots$

$$12x + 8y = 2^{11}3^7(8 + 3) = 11 \times 2^{11}3^7$$
$$6x + 8y = 2^{11}3^7(4 + 3) = 7 \times 2^{11}3^7$$

$$x^x y^y = 2^{12x+8y}3^{6x+8y} = 2^{11\times 2^{11}3^7}3^{7\times 2^{11}3^7} = \left(2^{11}\right)^{2^{11}3^7}\left(3^7\right)^{2^{11}3^7} = \left(2^{11}3^7\right)^{2^{11}3^7}$$

so that $z = 2^{11}3^7$