Contest 6

April 7, 1998

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$a = -18$.

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$| 1998 + 1999 - 2000 | = 1997$

Problem 3

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The perimeter is $1 + 1 + 1 + 1 + 2 = 5$

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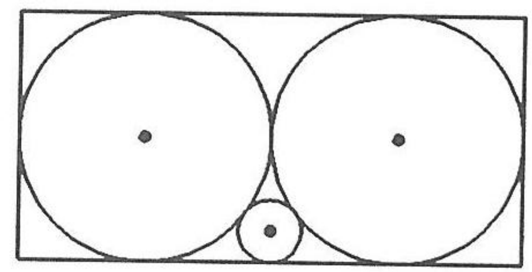
Two congruent circles are inscribed in a rectangle so that each is tangent to three sides of the rectangle and to the other circle. A third circle, smaller than the other two, is tangent to both congruent circles and one side of the rectangle. If the area of each congruent circle is 64π, what is the area of the small circle?



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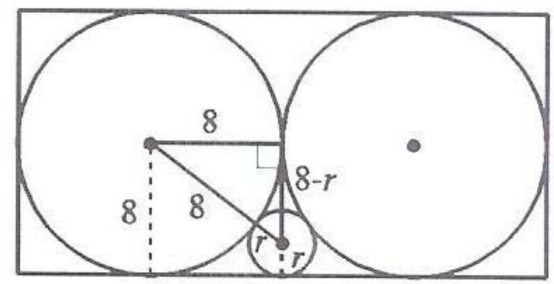

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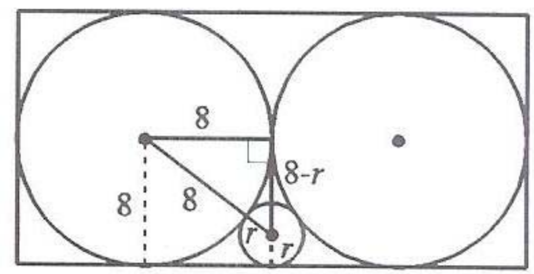

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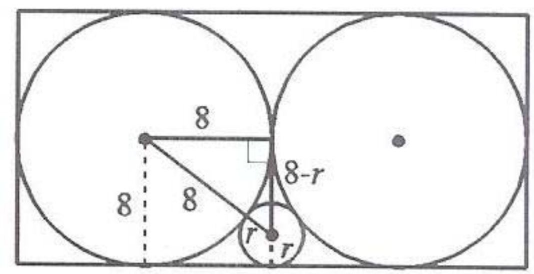

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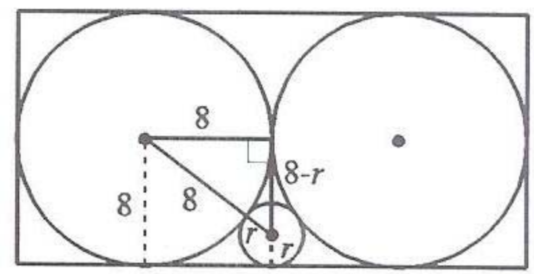

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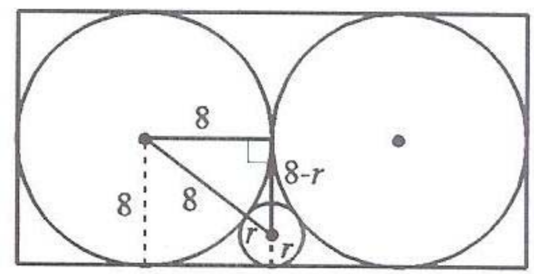

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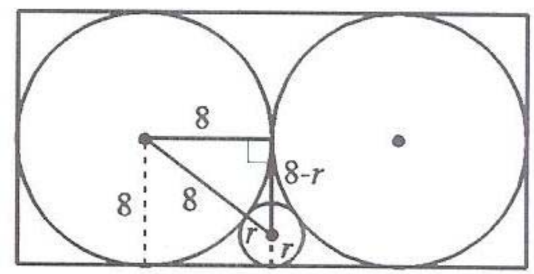

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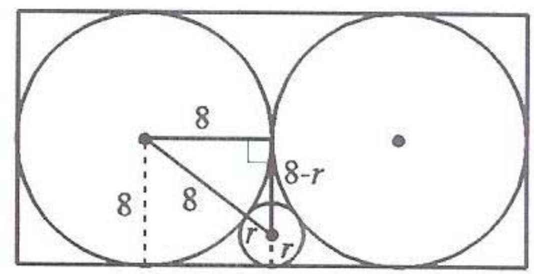

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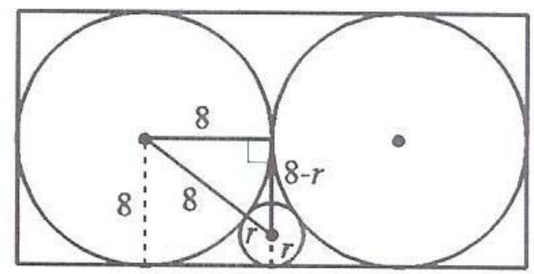

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$\tan A \tan C = 3$

NIM Problem From Last Week

In a certain two person game, each person, in turn, removes toothpicks from a common pile until the pile is exhausted. The person who takes the last toothpick wins.

Each turn, a player may take as many toothpicks as he/she wishes, but not more than was taken on the other player's last turn. To begin, the first person may take any number except all of the toothpicks.

If the starting pile contains a) 301, b) 302, c) 300, d) 304 toothpicks, how many toothpicks must the first player take on the first turn to guarantee a win with perfect subsequent play?

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What numbers on the starting pile guarantee the second person a win with perfect play?